Canad. Math. Bull. Vol. **29** (1), 1986

A PROBLEM ON RELATIVE PROJECTIVITY FOR ABELIAN GROUPS

BY

S. FEIGELSTOCK AND R. RAPHAEL

ABSTRACT. The article studies the class of abelian groups G such that in every direct sum decomposition $G = A \oplus B$, A is B-projective. Such groups are called pds groups and they properly include the quasi-projective groups.

The pds torsion groups are fully determined.

The torsion-free case depends on a lemma that establishes freedom in the non-indecomposable case for several classes of groups. There is evidence suggesting freedom in the general reduced torsion-free case but this is not established and prompts a logical discussion. It is shown, for example, that pds torsion-free groups must be Whitehead if they are not indecomposable, but that there exists Whitehead groups that are not pds if there exist non-free Whitehead groups.

The mixed case is characterized and examples are given.

Introduction. The purpose of this article is to study a class of abelian groups which we call pds groups. These groups arose by dualizing one formulation of a problem of Fuchs' considered in [2]. Although the class of pds groups is larger than the class of quasi-projective groups, the starting point of our study is the Fuchs-Rangaswamy classification of quasi-projective groups, obtained in [6].

We discuss in sequence, the torsion, the torsion free, and the mixed cases. The torsion pds groups need not be quasi-projective, but should a p-component fail to be, then it must be a single copy of $Z(p^\infty)$. A complete classification of the torsion pds-groups is obtained (Proposition 5).

The study of the torsion free case depends heavily on a result (Lemma 8) that is used to prove freedom in the non-indecomposable case for several classes of groups. There is strong evidence which leads one to suspect that the reduced torsion free pds-groups are precisely the reduced indecomposable torsion free groups, and the free groups; see Remark 15.

Unlike the case of quasi-projective groups, mixed pds groups do exist. They are described completely in Theorem 21.

Notation and terminology follows [4] and [5].

Received by the editors November 29, 1984 and, in revised form, January 30, 1985.

Key words: abelian group, projectivity, module.

AMS Subject Classification (1980): 13C10, 20K40.

This work was primarily done at Bar-llan University. The second author who was on leave from Concordia University is indebted to both institutions for their support. The authors are grateful for suggestions from L. Fuchs and B. Zimmerman.

DEFINITION 1. *A right R-module M is called pds (projective direct summands) if for every decomposition $M = N \oplus P$ into a direct sum of submodules, N is P-projective. This means that given R-module homomorphisms $f: N \to X$, $\eta : P \to X$, η onto, there is a homomorphism $g: N \to P$ so that $f = \eta g$. The notion of P-projectivity, and its dual P-injectivity were introduced by Azumaya, and were studied in* [1].

The following useful results are straightforward:

LEMMA 2.
(i) *A direct summand of a pds module is pds.*
(ii) *A quasi-projective module is pds.*
(iii) *If $M \oplus M$ is pds, then M is a quasi-projective module.*

From now on the only modules which will be considered are the Z-modules, i.e., abelian groups.

We shall require the following characterization of quasi-projective abelian groups.

THEOREM 3 [6]. *An abelian group is quasi-projective if and only if it is free, or if it is a torsion group with p-components of the form $\oplus_I Z(p^n)$ for some index set I, and n a fixed positive integer depending on p. There are no mixed quasi-projective groups.*

LEMMA 4. *Let G be pds, and let p be a prime such that $G_p \neq (0)$. Then $G_p \simeq Z(p^\infty)$ or $G_p \simeq \oplus_I Z(p^n)$ for some positive integer n, and index set I. In either case, G_p is a direct summand of G.*

PROOF. Once it has been shown that G_p has the desired form, then it will be a direct summand either because it is divisible, or because it is pure in G and bounded.

First suppose that G_p is not reduced. Then $G = Z(p^\infty) \oplus H$ for some subgroup H. If $H_p = (0)$ our result holds. Suppose that $H_p \neq (0)$. If H_p is not reduced then $Z(p^\infty)$ is a direct summand of H, and so by Lemma 2(i), $Z(p^\infty) \oplus Z(p^\infty)$ is pds which, by lemma 2(iii), implies that $Z(p^\infty)$ is quasi-projective. Theorem 3 yields a contradiction. Therefore H_p is reduced, and so H has a direct summand $Z(p^n)$, n a positive integer. Again Lemma 2(i) yields that $Z(p^\infty) \oplus Zp^n)$ is pds. The diagram

$$\begin{array}{c} Z(p^n) \\ \downarrow \\ Z(p^\infty) \leftarrow Z(p^\infty) \end{array}$$

with vertical map inclusion and horizontal map multiplication by p^n cannot be completed, a contradiction. Hence $H_p = (0)$.

Now suppose that G_p is reduced, and let B be a basic subgroup of G_p. Suppose that B has a direct summand $Z(p^n) \oplus Z(p^m)$ with $n < m$. Since this group is a bounded pure

https://doi.org/10.4153/CMB-1986-021-9 Published online by Cambridge University Press

subgroup of G, it is a direct summand of G, and therefore pds by Lemma 2(i). However, the diagram

$$\begin{array}{c} Z(p^n) \\ \downarrow \\ Z(p^n) \leftarrow Z(p^m) \end{array}$$

with vertical map the identity, and horizontal map induced by sending a generator of $Z(p^m)$ into a generator of $z(p^n)$ cannot be completed. This contradiction yields that $B = \bigoplus_I Z(p^n)$ n a fixed positive integer, and I an index set. Since B is a pure bounded basic subgroup of G_p, $G_p = B \oplus D$, with D divisible. However, G_p is reduced, and so $G_p = B$.

PROPOSITION 5. *Let G be a torsion group. G is pds if and only if each p-component of G has the form $Z(p^\infty)$ or $\bigoplus_I Z(p^n)$.*

PROOF. Lemma 4 shows the necessity of the form. For the sufficiency, one notes that G will be pds provided each G_p is because decompositions and maps respect the splitting of G into its p-components. $Z(p^\infty)$ is indecomposable, and $\bigoplus_I Z(p^n)$ is quasi-projective, so in either case the pds property holds.

We proceed to the torsion free case.

LEMMA 6. *An infinite rank torsion free group G has $Z(p^\infty)$ as a homomorphic image for every prime p.*

PROOF. G has an infinite rank free subgroup which maps onto $Z(p^\infty)$. This map extends to G by the injectivity of $Z(p^\infty)$.

LEMMA 7. *Let G be a torsion free pds group. Then either G is reduced, or $G \simeq Q \oplus H$ where H is a finite rank reduced group that has no $Z(p^\infty)$ as a homomorphic image.*

PROOF. If G is not reduced it has the form $G = Q \oplus H$ by the injectivity of Q. Since Q is not quasi-projective, $Q \oplus Q$ cannot be a direct summand of G, and so H is reduced. Suppose there exists a surjection $\eta : H \to Z(p^\infty)$, and consider the diagram

$$\begin{array}{c} Q \\ \downarrow \\ Z(p^\infty) \underset{\eta}{\leftarrow} H \end{array}$$

with vertical map a projection of Q onto $Z(p^\infty)$. A map $f: Q \to H$ closing the diagram must be nonzero and yields that the reduced group H possesses a divisible subgroup $f(Q)$. This contradiction implies that H has no $Z(p^\infty)$ homomorphic image and so has finite rank by Lemma 6.

LEMMA 8. *Let G be a torsion free pds group. Then either G is indecomposable, or every reduced direct summand of G has a nonzero free direct summand.*

PROOF. Suppose that $G = H \oplus K$ with $H \neq (0)$, $K \neq (0)$, and H reduced. For some prime p, $pH \neq H$, so there is a surjection $f: H \to H/pH \to Z(p)$.

Let $a \in K$, $a \neq 0$. Then $\bar{a}$ is an element of order p in $K/(pa)$. Let $\alpha: Z(p) \to (\bar{a})$ be an isomorphism, and let $\eta: K \to K/(pa)$ be the canonical map. Since G is pds there exists a map $\psi: H \to K$ completing the diagram

$$\begin{array}{c} H \\ f \downarrow \\ Z(p) \\ \alpha \downarrow \\ K/(pa) \underset{\eta}{\leftarrow} K. \end{array}$$

Then

$$Z(p) \simeq \frac{\psi(H) + \ker \eta}{\ker \eta} = \frac{\psi(H) + (pa)}{(pa)}.$$

By a theorem of Fuchs, Mostowski, and Sasiada [4, 18.3] $\psi(H) + (pa)$ is a direct sum of cyclic groups, and hence free. Therefore $\psi(H)$ is a nonzero free group, a fact which yields the result.

COROLLARY 9. *Let G be a torsion free group which is not reduced. Then G is pds if and only if $G \simeq Q \oplus F$ with F a finite rank free group.*

PROOF. Let G be pds. By Lemma 7, $G = Q \oplus H$ with H a finite rank reduced group. Lemma 8, and induction on the rank of H yield that H is free.

The converse follows from the projectivity of free groups, and the fact that no nonzero homomorphic image of Q is finitely generated.

COROLLARY 10. *Let G be a reduced torsion free group, and a direct sum of indecomposable groups. Then G is pds if and only if G is indecomposable or free.*

COROLLARY 11. *The completely decomposable torsion free pds groups are the rank 1 torsion free groups, the free groups, and groups of the form $Q \oplus F$, F finite rank free.*

LEMMA 12. *Let G be a reduced torsion free pds group that is not indecomposable. Then G is isomorphic to a subgroup of $\Pi_{|G|} Z$.*

PROOF. Let $G = H \oplus K$, $H \neq (0)$, $K \neq (0)$. It suffices to show that given $a \in H$, $a \neq 0$, there is a map $\psi: H \to Z$ with $\psi(a) \neq 0$. By Lemma 8 and 2(i), $H \oplus Z$ is a pds group. The set of elements in H with infinite p-height for all p constitutes a divisible subgroup of H. Since H is reduced, every nonzero element in H has finite p-height for some p and so there exists a positive integer n such that $a \in p^{n-1}H$ but $a \notin p^n H$. Therefore there is a map $\varphi: H \to H/p^n H \to Z(p^n)$ with $\varphi(a) \neq 0$. Let $\eta: Z \to Z(p^n)$ be the natural map. Then the map $\psi: H \to Z$ completing the diagram

$$\begin{array}{ccc} & H & \\ & \varphi \downarrow & \\ & Z(p^n) \underset{\eta}{\leftarrow} Z & \end{array}$$

satisfies $\psi(a) \neq 0$.

COROLLARY 13. *A countable reduced torsion free group is pds if and only if it is either indecomposable or free. If a reduced torsion free group is not indecomposable, then all of its countable subgroups are free.*

PROOF. A theorem of Baer [4, 19.2] states that all countable subgroups of products of Z are free.

COROLLARY 14. *Let $G = \bigoplus_{i \in I} G_i$ with G_i a reduced countable torsion free group for each $i \in I$. Then G is pds if and only if G is indecomposable or free.*

PROOF. Suppose that G is pds but not indecomposable. By Lemma 8 and Corollary 13 each G_i is free, and so G is free. The converse is obvious.

REMARK 15. If there exists a reduced torsion free pds group G which is not indecomposable, and not free, it must satisfy the following properties:

(a) By Corollaries 10 and 14, G is neither a direct sum of indecomposable groups, nor a direct sum of countable groups. By Lemma 8, the only possible indecomposable direct summand of G is Z.

(b) $G \leq \Pi_{|G|} Z$ and all countable subgroups of G are free.

(c) By Lemma 8, G has free direct summands of any finite rank. G cannot have a free direct summand of rank G, because if so, we have $G = F \oplus X$, F free, and a surjection from F onto X. The pds property implies that X (and hence G) is free.

(d) If $G = \bigoplus_J G_j$, then (c) and Lemma 8 imply that $|J| <$ rank G.

(e) Azumaya, Mbuntum and Varadarajan have shown [1, 1.7], that pure finite rank subgroups of G are free, and are direct summands of G. It follows by [5, p. 122, Exercise 2] that G is separable, homogeneous, and is a pure subgroup of $\Pi_{|G|} Z$, [5, 87.4].

The solution of the mixed case depends on determining which torsion free groups are D-projective, for D a torsion divisible group. The following facts from [1] are required:

DEFINITION 16 [1, 1.12]. *Let A, B, M be R-modules, and $\theta : A \to B$ an epimorphism. θ is said to be an M-epimorphism if there exists a map $\psi : A \to M$ satisfying* $\ker \theta \cap \ker \psi = (0)$.

PROPOSITION 17 [1, 1.13]. *A module B is M-projective if and only if every M-epimorphism $\theta : A \to B$ splits.*

THEOREM 18. *Let H be a torsion free group, and let D be a divisible torsion group. H is D-projective if and only if* $\operatorname{Ext}(H, T) = (0)$ *for every subgroup $T \leq D$.*

https://doi.org/10.4153/CMB-1986-021-9 Published online by Cambridge University Press

PROOF. Let $\theta : A \to H$ be a D-epimorphism, and let $\psi : A \to D$ be a map satisfying $\ker \theta \cap \ker \psi = (0)$. Then $\ker \theta \simeq \psi(\ker \theta) \leq D$. Therefore if Ext $(H, T) = (0)$ for every $T \leq D$ then θ splits, and H is D-projective by Proposition 17. Conversely, suppose there exists $T \leq D$ such that Ext $(H, T) \neq (0)$. Then there exists a group G such that $G_t \leq D$, $G/G_t \cong H$, but G_t is not a direct summand of G. Let $\theta : G \to G/G_t$ be the natural epimorphism, and let $\psi : G \to D$ be an extension of the inclusion map $G_t \to D$: the injectivity of D assures the existence of ψ. Since $\ker \theta \cap \ker \psi = (0)$ we have that θ is a D-epimorphism. However θ does not split, so by Proposition 17, H is not D-projective.

COROLLARY 19. *Let P be a finite set of primes. Then every torsion free group H is $\bigoplus_{p \in P} Z(p^\infty)$-projective.*

PROOF. Every subgroup $T \leq \bigoplus_{p \in P} Z(p^\infty)$ is the direct sum of a bounded group and a divisible group, so by [5, 100.1] Ext $(H, T) = 0$ for every torsion free group H. Therefore Theorem 18 yields that H is $\bigoplus_{p \in P} Z(p^\infty)$-projective.

Actually, it can be shown that a group G is $\bigoplus_{p \in P} Z(p^\infty)$-projective, P a finite set of primes, if and only if $G_p = (0)$ for all $p \in P$.

LEMMA 20. *Let G be a mixed pds group, and let p be a prime for which $G_p \neq (0)$. Then $G = G_p \oplus K$ with K a group which has no nonzero torsion free elements with infinite p-height.*

PROOF. Lemma 4 assures the existence of a subgroup $K \leq G$ such that $G = G_p \oplus K$. Suppose there exists a torsion free element $a \in K$, $a \neq 0$, with infinite p-height. Put $a_1 = a$, and inductively choose $a_{n+1} \in K$ such that $pa_{n+1} = a_n$ for every positive integer n. Then $\cup(\bar{a}_n)$ is a subgroup of $K/(pa_1)$ isomorphic to $Z(p^\infty)$ and so $K/(pa_1)$ has direct summand $Z(p^\infty)$. Therefore there is an epimorphism $\eta : K \to Z(p^\infty)$. Now there exists a non-zero map $f : G_p \to Z(p^\infty)$. Since K has no nonzero p-elements the diagram

$$\begin{array}{ccc} G_p & & \\ f \downarrow & & \\ Z(p^\infty) & \underset{\eta}{\leftarrow} & K \end{array}$$

cannot be closed, a contradiction.

THEOREM 21. *Let G be a mixed group. G is pds if and only if $G = \bigoplus_{p \in P} Z(p^\infty) \oplus H$ with P a set of distinct primes, and H a finite rank torsion free group which is either free or indecomposable and satisfies the following two properties:* (1) *$Z(p^\infty)$ is not a homomorphic image of H for all $p \in P$,* (2) Ext $(H, T) = 0$ *for every subgroup $T \leq \bigoplus_{p \in P} Z(p^\infty)$.*

PROOF. Suppose that G is a mixed pds-group, and let p be a prime for which $G_p \neq 0$. By Lemma 4, $G = G_p \oplus K$, and either $G_p \simeq Z(p^\infty)$ or is the direct sum of copies of $Z(p^n)$ for some positive integer n. Suppose the latter. Then there is a surjection

https://doi.org/10.4153/CMB-1986-021-9 Published online by Cambridge University Press

$f: G_p \to Z(p^n)$, and by Lemma 20, there is an epimorphism $\eta: K \to K/p^nK \to Z(p^n)$. Since K has no nonzero p-elements, the diagram

$$\begin{array}{c} G_p \\ f \downarrow \\ Z(p^n) \underset{\eta}{\leftarrow} K \end{array}$$

cannot be completed, a contradiction.

Therefore $G_p \simeq Z(p^\infty)$ and G_t has the form $\bigoplus_{p \in P} Z(p^\infty)$, P a set of distinct primes. Suppose there is an epimorphism $\eta: H \to Z(p^\infty)$ for some $p \in P$. By Lemma 2(i), $H \oplus Z(p^\infty)$ is pds, but the diagram

$$\begin{array}{c} Z(p^\infty) \\ \downarrow \\ Z(p^\infty) \underset{\eta}{\leftarrow} H \end{array}$$

with vertical map the identity, cannot be completed, a contradiction. H has finite rank by Lemma 6, and is either free or indecomposable by Corollary 13.

Conversely, let $G = \bigoplus_{p \in P} Z(p^\infty) \oplus H$ with P and H satisfying the conditions of the theorem. It is readily seen because of condition (i), that it suffices to complete every diagram

$$\begin{array}{c} H \\ f \downarrow \\ L \underset{\eta}{\leftarrow} \bigoplus_{p \in P} Z(p^\infty) \end{array}$$

with η an epimorphism, i.e., to prove that H is $\bigoplus_{p \in P} Z(p^\infty)$-projective. Theorem 18 assures that this is indeed the case.

Observe that if the set of primes P is finite, then condition (2) of Theorem 21 is superfluous by Corollary 19.

REMARK 22. Conditions (1) and (2) of Theorem 21 are independent. If $G = \bigoplus_{p \in P} Z(p^\infty) \oplus H$ where P is the set of all primes, and H is the subgroup of Q generated by $\{1/p, p \in P\}$, then no $Z(p^\infty)$ is a homomorphic image of H. If $T = \bigoplus_{p \in P} Z(p)$, then Ext $(H, T) \neq (0)$ because condition (a) of [5, p. 193, Ex. 6] fails, $-(0)$ is a finite rank pure subgroup of H and $1 \in H$ is divisible by all $p \in P$. This shows that (1) $\not\Rightarrow$ (2). Conversely, consider $Z(2^\infty) \oplus Q$. $Z(2^\infty)$ is a homorphic image of Q but (2) holds because the subgroups T of $Z(2^\infty)$ are either bounded or divisible, whence Ext $(Q, T) = (0)$.

COROLLARY 23. *A mixed group G is pds if and only if $G = \bigoplus_{p \in P} Z(p^\infty) \oplus H$ with P a set of distinct primes, and H a finite rank reduced torsion-free group which is either free or indecomposable and satisfies* (i) *for all $p \in P$, $Z(p^\infty)$ is not a homomorphic image of H, and* (ii) *for S a pure subgroup of H, no nonzero element of H/S is divisible by all $p \in P$.*

Proof. This again uses Exercises 6 and 7 of [5, p. 193–194]. (Condition (b) of exercise 6 is vacuously satisfied).

Corollary 24. *Let H be rank one torsion-free and let P be a set of distinct primes. The group* $\bigoplus_{p\in P} Z(p^\infty) \oplus H$ *is pds if and only if the type of H,* $t(H)$ *has finite p-component* $t_p(H)$ *for every* $p \in P$. *If P is an infinite set then* $t_p(H)$ *equals zero for infinitely many* $p \in P$.

Example. *If H is a rank one torsion-free group then* $H \oplus Q/Z$ *is pds if and only if* $t(H) = (k_1, \ldots\ldots, k_n, \ldots\ldots)$ *where all* k_n *are finite and infinitely many* k_n *equal zero.*

Fuchs has pointed out that for any rank $r \geq 2$, there exists an indecomposable torsion-free group of type $(0, 0, \ldots)$ having $Z(p^\infty)$ as a homomorphic image (c.f. 5, p. 125).

We close with a discussion relating the pds problem with Whitehead groups.

Proposition 25. *Let G be pds, torsion-free, and let it have the form* $H \oplus K$, *where H is of infinite rank. Then K is Whitehead.*

Proof. K is H-projective so it is Q-projective because Q is a homomorphic image of K [1, Prop. 1.16(1)]. Now the divisibility of Q and the argument used in the second half of the proof of Theorem 18 show that Ext $(K, T) = (0)$ for all subgroups T of Q. Thus Ext $(K, Z) = (0)$ and K is Whitehead.

Corollary 26. *Suppose that G is torsion-free, reduced, pds and has an infinite direct sum decomposition. Then G is Whitehead.*

Proof. Let $G = \bigoplus_I G_i$ with $|I|$ infinite. By Lemma 8 each $G_i \simeq Z \oplus H_i$ so $G \simeq (\bigoplus_I Z) \oplus (\bigoplus_I H_i)$. By the proposition $\bigoplus_I H_i$ is Whitehead and therefore G is as well.

Shelah has shown that the freedom of Whitehead groups is undecidable under Zermelo-Frankel and the continuum hypothesis. Freedom does follow from the Godel axiom of constructability ($V = L$) so to the discussion in Remark 15 one can add:

Corollary 27. *Suppose that* $V = L$. *Then a reduced torsion-free pds group that is not free and not indecomposable has only finitely indexed direct sum decompositions.*

B. Zimmerman has remarked that the same result holds if one only assumes that the decompositions are indexed by non-measurable cardinals.

Proposition 28. *If every Whitehad group is pds, then every Whitehead group is free.*

Proof. Let G be a Whitehead group, and let $\varphi : A \to B$ be an epimorphism between abelian groups, and $\psi : G \to B$ a homomorphism. There exists a free group F and an epimorphism $\rho : F \to A$. We therefore have:

$$\begin{array}{ccccc} G & & & & \\ \psi \downarrow & & & & \\ B & \underset{\varphi}{\leftarrow} & A & \underset{\rho}{\leftarrow} & F \end{array}$$

https://doi.org/10.4153/CMB-1986-021-9 Published online by Cambridge University Press

Since direct sums of Whitehead groups are Whitehead, [4, p. 179(c)], $G \oplus F$ is a Whitehead group, and hence pds. Therefore there exists a homomorphism $\mu : G \to F$ such that $\varphi\rho\mu = \psi$. This implies that G is projective, and therefore free [3, Theorem 14.6].

Let $W \to$ pds denote the problem of determining whether or not every Whitehead group is pds. Proposition 28, together with celebrated results of S. Shelah, yields the following:

COROLLARY 29. $W \to$ pds *is undecidable in Zermelo-Frankel set theory + continuum hypothesis, true in Zermelo-Frankel set theory + "$V = L$", and false in Zermelo-Frankel set theory + Martin's axiom + negation of the continuum hypothesis.*

REMARK 30. Let G be a W-group that is not free. Then for every free group F such that $|F| \geqslant |G|$, $G \oplus F$ is not pds.

PROOF. G is an epimorphic image of F.

This remark shows that under Zermelo Fraenkel, Martin's Axiom, and the negation of the continuum hypothesis, there exist Whitehead groups of arbitrarily large cardinality that are not pds.

REFERENCES

1. G. Azumaya, F. Mbuntun, and K. Varadarajan, *On M-projective and M-injective modules*, Pac. J. Math. **59** (1975), pp. 9–16.

2. W. Burgess and R. Raphael, *Sur les modules à facteurs directs absolus*, (submitted).

3. L. Fuchs, *Abelian Groups*, Pergamon Press, N.Y., 1960.

4. L. Fuchs, *Infinite Abelian Groups*, Vol. I, Academic Press, New York–London, 1971.

5. L. Fuchs, *Infinite Abelian Groups*, Vol. II, Academic Press, New York–London, 1973.

6. L. Fuchs and K. Rangaswamy, *Quasi-projective abelian groups*, Bull Soc. Math., France **98** (1970), pp. 5–8.

BAR-ILAN UNIVERSITY
RAMAT-GAN, ISRAEL

CONCORDIA UNIVERSITY
MONTREAL, CANADA