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# SECOND HANKEL DETERMINANT FOR LOGARITHMIC INVERSE COEFFICIENTS OF CONVEX AND STARLIKE FUNCTIONS

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**Abstract**

We obtain sharp bounds for the second Hankel determinant of logarithmic inverse coefficients for starlike and convex functions.

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## 1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Here $\mathcal{H}$ is a locally convex topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Let $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$ such that $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions which are univalent (that is, one-to-one) in $\mathbb{D}$. If $f \in \mathcal{A}$, then it has the series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of the Taylor coefficients of the function $f \in \mathcal{A}$ of the form (1.1) is

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

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Hankel determinants of various orders have been studied in many contexts (see for instance [5]). The Fekete–Szegö functional is the second Hankel determinant $H_{2,1}(f)$. Fekete–Szegö obtained estimates for $|a_3 - \mu a_2^2|$ with $\mu$ real (see [10, Theorem 3.8]).

Let $g$ be the inverse function of $f \in \mathcal{S}$ defined in a neighbourhood of the origin with the Taylor series expansion

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n, \tag{1.2}$$

where we may choose $|w| < 1/4$ from Koebe's 1/4-theorem. Using variational methods, Löwner [16] obtained the sharp estimate

$$|A_n| \le K_n \quad \text{for each } n \in \mathbb{N},$$

where $K_n = (2n)!/(n!(n+1)!)$ and $K(w) = w + K_2 w^2 + K_3 w^3 + \cdots$ is the inverse of the Koebe function. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a function in class $\mathcal{S}$. Since $f(f^{-1})(w) = w$, it follows from (1.2) that

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= -a_3 + 2a_2^2, \\ A_4 &= -a_4 + 5a_2 a_3 - 5a_2^3. \end{aligned}$$

The *logarithmic coefficients* $\gamma_n$ of $f \in \mathcal{S}$ are defined by

$$F_f(z) := \log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D}. \tag{1.3}$$

Few exact upper bounds for $\gamma_n$ have been established. The significance of this problem in the context of the Bieberbach conjecture was pointed out by Milin [17]. Milin's conjecture that for $f \in \mathcal{S}$ and $n \ge 2$,

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left( k|\gamma_k|^2 - \frac{1}{k} \right) \le 0,$$

led De Branges, by proving this conjecture, to the proof of the Bieberbach conjecture [9]. For the Koebe function $k(z) = z/(1-z)^2$, the logarithmic coefficients are $\gamma_n = 1/n$. Since the Koebe function $k$ plays the role of extremal function for most of the extremal problems in the class $\mathcal{S}$, it might be expected that $|\gamma_n| \le 1/n$ holds for functions in $\mathcal{S}$. However, this is not true in general, even in order of magnitude. Indeed, there exists a bounded function $f$ in the class $\mathcal{S}$ with logarithmic coefficients $\gamma_n \ne O(n^{-0.83})$ (see [10, Theorem 8.4]). By differentiating (1.3) and equating coefficients,

$$\begin{aligned} \gamma_1 &= \tfrac{1}{2} a_2, \\ \gamma_2 &= \tfrac{1}{2}\left(a_3 - \tfrac{1}{2} a_2^2\right), \\ \gamma_3 &= \tfrac{1}{2}\left(a_4 - a_2 a_3 + \tfrac{1}{3} a_2^3\right). \end{aligned} \tag{1.4}$$

If $f \in \mathcal{S}$, it is easy to see that $|\gamma_1| \le 1$, because $|a_2| \le 2$. Using the Fekete–Szegö inequality [10, Theorem 3.8] for functions in $\mathcal{S}$ in (1.3), we obtain the sharp estimate

$$|\gamma_2| \le \tfrac{1}{2}(1 + 2e^{-2}) = 0.635\ldots.$$

For $n \ge 3$, the problem seems much harder and no significant bound for $|\gamma_n|$ when $f \in \mathcal{S}$ appears to be known. In 2017, Ali and Allu [1] obtained initial logarithmic coefficient bounds for close-to-convex functions. For recent results on several subclasses of close-to-convex functions, see [2, 6, 21].

The notion of logarithmic inverse coefficients, that is, logarithmic coefficients of the inverse of $f$, was proposed by Ponnusamy *et al.* [20]. The *logarithmic inverse coefficients* $\Gamma_n$, $n \in \mathbb{N}$, of $f$ are defined by the equation

$$F_{f^{-1}}(w) := \log \frac{f^{-1}(w)}{w} = 2\sum_{n=1}^{\infty} \Gamma_n w^n, \quad |w| < 1/4.$$

In [20], Ponnusamy *et al.* found sharp upper bounds for the logarithmic inverse coefficients for the class $\mathcal{S}$, namely

$$|\Gamma_n| \le \frac{1}{2n}\binom{2n}{n}, \quad n \in \mathbb{N},$$

with equality only for the Koebe function or one of its rotations. Ponnusamy *et al.* [20] also obtained sharp bounds for the initial logarithmic inverse coefficients for some of the important geometric subclasses of $\mathcal{S}$.

Recently, Kowalczyk and Lecko [12] proposed the study of the Hankel determinant whose entries are logarithmic coefficients of $f \in \mathcal{S}$, given by

$$H_{q,n}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

Kowalczyk and Lecko [12] obtained a sharp bound for the second Hankel determinant $H_{2,1}(F_f/2)$ for starlike and convex functions. Sharp bounds for $H_{2,1}(F_f/2)$ for various subclasses of $\mathcal{S}$ are considered in [3, 4, 11, 13, 18]).

In this paper, we consider the second Hankel determinant for logarithmic inverse coefficients. From (1.4), for $f \in \mathcal{S}$ given by (1.1), the second Hankel determinant of $F_{f^{-1}}/2$ is given by

$$\begin{aligned} H_{2,1}(F_{f^{-1}}/2) = \Gamma_1\Gamma_3 - \Gamma_2^2 &= \tfrac{1}{4}(A_2A_4 - A_3^2 + \tfrac{1}{4}A_2^4) \\ &= \tfrac{1}{48}(13a_2^4 - 12a_2^2a3 - 12a_3^2 + 12a_2a_4). \end{aligned} \tag{1.5}$$

We note that $|H_{2,1}(F_{f^{-1}}/2)|$ is invariant under rotation, since for $f_\theta(z) := e^{-i\theta} f(e^{i\theta}z)$, $\theta \in \mathbb{R}$ and $f \in \mathcal{S}$,

$$H_{2,1}(F_{f_\theta^{-1}}/2) = \frac{e^{4i\theta}}{48}(13a_2^4 - 12a_2^2a3 - 12a_3^2 + 12a_2a_4) = e^{4i\theta} H_{2,1}(F_{f^{-1}}/2).$$

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The main aim of this paper is to find a sharp upper bound for $|H_{2,1}(F_{f^{-1}}/2)|$ when $f$ belongs to the class of convex or starlike functions. A domain $\Omega \subseteq \mathbb{C}$ is said to be starlike with respect to a point $z_0 \in \Omega$ if the line segment joining $z_0$ to any point in $\Omega$ lies entirely in $\Omega$. If $z_0$ is the origin, then we say that $\Omega$ is a starlike domain. A function $f \in \mathcal{A}$ is said to be starlike if $f(\mathbb{D})$ is a starlike domain. We denote by $\mathcal{S}^*$ the class of starlike functions $f$ in $\mathcal{S}$. It is well known that a function $f \in \mathcal{A}$ is in $\mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}. \tag{1.6}$$

Further, a domain $\Omega \subseteq \mathbb{C}$ is called convex if the line segment joining any two points of $\Omega$ lies entirely in $\Omega$. A function $f \in \mathcal{A}$ is called convex if $f(\mathbb{D})$ is a convex domain. We denote by $\mathcal{C}$ the class of convex functions in $\mathcal{S}$. A function $f \in \mathcal{A}$ is in $\mathcal{C}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}. \tag{1.7}$$

## 2. Preliminary results

In this section, we present the key lemmas which will be used to prove the main results of this paper. Let $\mathcal{P}$ denote the class of all analytic functions $p$ having positive real part in $\mathbb{D}$, with the form

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots . \tag{2.1}$$

A member of $\mathcal{P}$ is called a Carathéodory function. It is known that $|c_n| \le 2, n \ge 1$, for $p \in \mathcal{P}$. By using (1.6) and (1.7), functions in the classes $\mathcal{S}^*$ and $\mathcal{C}$ can be represented in terms of functions in the Carathéodory class $\mathcal{P}$.

Parametric representations of the coefficients are often useful. In Lemma 2.1, (2.2) is due to Carathéodory [10]. Equation (2.3) can be found in [19]. In 1982, Libera and Zlotkiewicz [14, 15] derived (2.4) with the assumption that $c_1 \ge 0$. Later, Cho *et al.* [7] derived (2.4) in the general case and gave the explicit form of the extremal function.

LEMMA 2.1. *If $p \in \mathcal{P}$ is of the form (2.1), then*

$$c_1 = 2p_1, \tag{2.2}$$

$$c_2 = 2p_1^2 + 2(1 - p_1^2)p_2 \tag{2.3}$$

*and*

$$c_3 = 2p_1^3 + 4(1 - p_1^2)p_1 p_2 - 2(1 - p_1^2)p_1 p_2^2 + 2(1 - p_1^2)(1 - |p_2|^2)p_3 \tag{2.4}$$

*for some $p_1, p_2, p_3 \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$.*

*For $p_1 \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, there is a unique function $p \in \mathcal{P}$ with $c_1$ as in (2.2), namely*

$$p(z) = \frac{1 + p_1 z}{1 - p_1 z}, \quad z \in \mathbb{D}.$$

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*For $p_1 \in \mathbb{D}$ and $p_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with $c_1$ and $c_2$ as in (2.2) and (2.3), namely*

$$p(z) = \frac{1 + (p_1 + \overline{p_1}p_2)z + p_2z^2}{1 - (p_1 - \overline{p_1}p_2)z - p_2z^2}. \tag{2.5}$$

*For $p_1, p_2 \in \mathbb{D}$ and $p_3 \in \mathbb{T}$, there is unique function $p \in \mathcal{P}$ with $c_1, c_2$ and $c_3$ as in (2.2)–(2.4), namely*

$$p(z) = \frac{1 + (\overline{p_2}p_3 + \overline{p_1}p_2 + p_1)z + (\overline{p_1}p_3 + p_1\overline{p_2}p_3 + p_2)z^2 + p_3z^3}{1 + (\overline{p_2}p_3 + \overline{p_1}p_2 - p_1)z + (\overline{p_1}p_3 - p_1\overline{p_2}p_3 - p_2)z^2 - p_3z^3}, \quad z \in \mathbb{D}.$$

Next we recall the following well-known result due to Choi *et al.* [8].

LEMMA 2.2. *Let $A, B, C$ be real numbers and*

$$Y(A, B, C) := \max_{z\in\overline{\mathbb{D}}}(|A + Bz + Cz^2| + 1 - |z|^2).$$

(i) *If $AC \geq 0$, then*

$$Y(A, B, C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1 - |C|), \\ 1 + |A| + \dfrac{B^2}{4(1 - |C|)}, & |B| < 2(1 - |C|). \end{cases}$$

(ii) *If $AC < 0$, then*

$$Y(A, B, C) = \begin{cases} 1 - |A| + \dfrac{B^2}{4(1 - |C|)}, & -4AC(C^{-2} - 1) \leq B^2 \wedge |B| < 2(1 - |C|), \\ 1 + |A| + \dfrac{B^2}{4(1 + |C|)}, & B^2 < \min\{4(1 + |C|)^2, -4AC(C^{-2} - 1)\}, \\ R(A, B, C), & \textit{otherwise}, \end{cases}$$

*where*

$$R(A, B, C) = \begin{cases} |A| + |B| + |C|, & |C|(|B| + 4|A|) \leq |AB|, \\ -|A| + |B| + |C|, & |AB| \leq |C|(|B| - 4|A|), \\ (|A| + |C|)\sqrt{1 - \dfrac{B^2}{4AC}}, & \textit{otherwise}. \end{cases}$$

## 3. Main results

Now we will prove the first main result of this paper. We obtain the following sharp bound for $H_{2,1}(F_{f^{-1}}/2)$ for functions in the class $\mathcal{C}$.

THEOREM 3.1. *For $f \in \mathcal{C}$ given by (1.1),*

$$|H_{2,1}(F_{f^{-1}}/2)| \leq \tfrac{1}{33}. \tag{3.1}$$

*The inequality is sharp.*

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PROOF. Let $f \in C$ be of the form (1.1). Then by (1.7),

$$1 + \frac{zf''(z)}{f'(z)} = p(z) \tag{3.2}$$

for some $p \in \mathcal{P}$ of the form (2.1). Since the class $C$ is invariant under rotation and the function is also rotationally invariant, we can assume that $c_1 \in [0, 2]$. Comparing the coefficients on both sides of (3.2) yields

$$\begin{aligned} a_2 &= \tfrac{1}{2}c_1, \\ a_3 &= \tfrac{1}{6}(c_2 + c_1^2), \\ a_4 &= \tfrac{1}{24}(2c_3 + 3c_1c_2 + c_1^3). \end{aligned}$$

Hence, by (1.5),

$$H_{2,1}(F_{f^{-1}}/2) = \tfrac{1}{2304}(11c_1^4 - 20c_1^2c_2 - 16c_2^2 + 24c_1c_3).$$

By (2.2)–(2.4), after simplification,

$$\begin{aligned} H_{2,1}(F_{f^{-1}}/2) = \frac{p_1^4}{48} &- \frac{1}{24}(1-p_1^2)p_1^2p_2 - \frac{1}{72}(1-p_1^2)(2+p_1^2)p_2^2 \\ &+ \frac{1}{24}(1-p_1^2)(1-|p_1^2|)p_1p_3. \end{aligned} \tag{3.3}$$

We consider three cases according to the value of $p_1$.

*Case 1:* $p_1 = 1$. By (3.3),

$$|H_{2,1}(F_{f^{-1}}/2)| = \tfrac{1}{48}.$$

*Case 2:* $p_1 = 0$. By (3.3),

$$|H_{2,1}(F_{f^{-1}}/2)| = \tfrac{1}{36}|p_2^2| \le \tfrac{1}{36}.$$

*Case 3:* $p_1 \in (0, 1)$. Since $|p_3| \le 1$, applying the triangle inequality in (3.3) gives

$$\begin{aligned} |H_{2,1}(F_{f^{-1}}/2)| &= \frac{1}{24}p_1(1-p_1^2)\Big(\Big|\frac{p_1^3}{2(1-p_1^2)} - p_1p_2 - \frac{2+p_1^2}{3p_1}p_2^2\Big| + 1 - |p_2^2|\Big) \\ &\le \frac{1}{24}p_1(1-p_1^2)(|A + Bp_2 + Cp_2^2| + 1 - |p_2^2|), \end{aligned} \tag{3.4}$$

where

$$A := \frac{p_1^3}{2(1-p_1^2)}, \quad B := -p_1, \quad C := -\frac{2+p_1^2}{3p_1}.$$

Since $AC < 0$, we can apply Lemma 2.2(ii). The argument now divides into five parts.

3($a$). For $p_1 \in (0, 1)$,

$$-4AC\Big(\frac{1}{C^2} - 1\Big) - B^2 = -\frac{p_1^2(14 + p_1^2)}{3(2 + p_1^2)} \le 0.$$

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The inequality $|B| < 2(1-|C|)$ is equivalent to $p_1(4-6p_1+5p_1^2) < 0$ which is not true for $p_1 \in (0,1)$.

3(*b*). It is easy to check that

$$\min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2}-1\right)\right\} = -4AC\left(\frac{1}{C^2}-1\right),$$

and from 3(*a*),

$$-4AC\left(\frac{1}{C^2}-1\right) \le B^2.$$

Therefore, the inequality $B^2 < \min\{4(1+|C|)^2, -4AC(1/C^2-1)\}$ does not hold for $0 < p_1 < 1$.

3(*c*). The inequality $|C|(|B|+4|A|)-|AB| \le 0$ is equivalent to $4+6p_1^2-p_1^4 \le 0$, which is false for $p_1 \in (0,1)$.

3(*d*). The inequality

$$|AB| - |C|(|B|-4|A|) = \frac{9p_1^4+10p_1^2-4}{1-p_1^2} \le 0$$

is equivalent to $9p_1^4+10p_1^2-4 \le 0$, which is true for

$$0 < p_1 \le p_1' = \tfrac{1}{3}\sqrt{\sqrt{61}-5} \approx 0.5588.$$

It follows from Lemma 2.2 and (3.4) that

$$|H_{2,1}(F_{f^{-1}}/2)| \le \tfrac{1}{24}p_1(1-p_1^2)(-|A|+|B|+|C|) = \tfrac{1}{144}(4+4p_1^2-11p_1^4) = h(p_1),$$

where $h(x) = 4+4x^2-11x^4$. By a simple calculation, the maximum of the function $h(x)$ for $0 < x \le p_1'$ occurs at the point $x_0 = \sqrt{2/11}$. We conclude that

$$|H_{2,1}(F_{f^{-1}}/2)| \le h\left(\sqrt{\tfrac{2}{11}}\right) = \tfrac{1}{33}.$$

3(*e*). For $p_1' < p_1 < 1$, we use the last case of Lemma 2.2 together with (3.4) to obtain

$$\begin{aligned}|H_{2,1}(F_{f^{-1}}/2)| &\le \frac{1}{24}p_1(1-p_1^2)(|C|+|A|)\sqrt{1-\frac{B^2}{4AC}}\\ &= \frac{1}{144}(p_1^4-2p_1^2+4)\sqrt{\frac{7-p_1^2}{4+2p_1^2}} = k(p_1),\end{aligned}$$

where

$$k(x) = \frac{1}{144}(x^4-2x^2+4)\sqrt{\frac{7-x^2}{4+2x^2}}.$$

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We want to find the maximum of $k(x)$ over the interval $p_1' < x < 1$. Observe that

$$k'(x) = \frac{x}{144}\sqrt{\frac{7-x^2}{4+2x^2}}\left(\frac{92-54x^2-15x^4+4x^6}{(-7+x^2)(2+x^2)}\right) = 0$$

if and only if $92 - 54x^2 - 15x^4 + 4x^6 = 0$. However, all the real roots of this equation lie outside the interval $p_1' < x < 1$ and $k'(x) < 0$ for $p_1' < x < 1$. So $k$ is decreasing and hence $k(x) \le k(p_1')$ for $p_1' < x < 1$. We conclude that, for $p_1' < x < 1$,

$$|H_{2,1}(F_{f^{-1}}/2)| \le k(p_1') \approx 0.0290035.$$

The desired inequality (3.1) follows from Cases 1–3. By tracking back in the proof, we see that equality in (3.1) holds when

$$p_1 = \sqrt{\tfrac{2}{11}}, \quad p_3 = 1,$$

and

$$|A + Bp_2 + Cp_2^2| + 1 - |p_2^2| = -|A| + |B| + |C|, \tag{3.5}$$

where

$$A = \tfrac{1}{9}\sqrt{\tfrac{2}{11}}, \quad B = -\sqrt{\tfrac{2}{11}}, \quad C = 4\sqrt{\tfrac{2}{11}}.$$

Indeed, we can easily verify that one of the solutions of (3.5) is $p_2 = 1$. In view of Lemma 2.2, we conclude that equality holds for the function $f \in \mathcal{A}$ given by (1.7), corresponding to the function $p \in \mathcal{P}$ of the form (2.5) with $p_1 = \sqrt{2/11}$, $p_2 = 1$ and $p_3 = 1$, that is,

$$p(z) = \frac{1 + 2\sqrt{2/11}z + z^2}{1 - z^2}.$$

This complete the proof. □

Next, we obtain the sharp bound for $H_{2,1}(F_{f^{-1}}/2)$ for functions in the class $\mathcal{S}^*$.

THEOREM 3.2. *For $f \in \mathcal{S}^*$ given by (1.1),*

$$|H_{2,1}(F_{f^{-1}}/2)| \le \tfrac{13}{12}. \tag{3.6}$$

*The inequality is sharp.*

PROOF. Let $f \in \mathcal{S}^*$ be of the form (1.1). By (1.6),

$$\frac{zf'(z)}{f(z)} = p(z) \tag{3.7}$$

for some $p \in \mathcal{P}$ of the form (2.1). By comparing the coefficients on both sides of (3.7),

$$\begin{aligned} a_2 &= c_1, \\ a_3 &= \tfrac{1}{2}(c_2 + c_1^2), \\ a_4 &= \tfrac{1}{6}(2c_3 + 3c_1c_2 + c_1^3). \end{aligned}$$

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Hence, by (1.5),

$$H_{2,1}(F_{f^{-1}}/2) = \tfrac{1}{48}(6c_1^4 - 6c_1^2c_2 - 3c_2^2 + 4c_1c_3).$$

From (2.2)–(2.4), by straightforward computation,

$$\begin{aligned} H_{2,1}(F_{f^{-1}}/2) = \tfrac{13}{12}p_1^4 - \tfrac{5}{2}(1-p_1^2)p_2 - \tfrac{1}{12}(1-p_1^2)(3+p_1^2)p_2^2 \\ + \tfrac{1}{3}(1-p_1^2)(1-|p_1^2|)p_1p_3. \end{aligned} \tag{3.8}$$

Now we consider three cases according to the value of $p_1$.

*Case 1:* $p_1 = 1$. By (3.8),

$$|H_{2,1}(F_{f^{-1}}/2)| = \tfrac{13}{12}.$$

*Case 2:* $p_1 = 0$. By (3.8),

$$|H_{2,1}(F_{f^{-1}}/2)| = \tfrac{1}{4}|p_2^2| \le \tfrac{1}{4}.$$

*Case 3:* $p_1 \in (0,1)$. Applying the triangle inequality in (3.8) and using the fact that $|p_3| \le 1$,

$$\begin{aligned} |H_{2,1}(F_{f^{-1}}/2)| &= \frac{1}{3}p_1(1-p_1^2)\Big(\Big|\frac{13p_1^3}{4(1-p_1^2)} - \frac{5}{2}p_1p_2 - \frac{3+p_1^2}{4p_1}p_2^2\Big| + 1 - |p_2^2|\Big) \\ &\le \frac{1}{24}p_1(1-p_1^2)(|A + Bp_2 + Cp_2^2| + 1 - |p_2^2|), \end{aligned}$$

where

$$A := \frac{13p_1^3}{4(1-p_1^2)}, \quad B := -\frac{5}{2}p_1, \quad C := -\frac{3+p_1^2}{4p_1}.$$

Since $AC < 0$, we can apply Lemma 2.2(ii).

3(*a*). For $p_1 \in (0,1)$,

$$-4AC\Big(\frac{1}{C^2} - 1\Big) - B^2 = -\frac{3p_1^2(16+p_1^2)}{(3+p_1^2)} \le 0.$$

The inequality $|B| < 2(1-|C|)$ is equivalent to $3 - 4p_1 + 2p_1^2 < 0$ which is not true for $p_1 \in (0,1)$.

3(*b*). It is easy to see that

$$\min\Big\{4(1+|C|)^2, -4AC\Big(\frac{1}{C^2} - 1\Big)\Big\} = -4AC\Big(\frac{1}{C^2} - 1\Big),$$

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and from 3($a$),

$$-4AC\Big(\frac{1}{C^2}-1\Big) \le B^2.$$

Therefore, the inequality $B^2 < \min\{4(1+|C|)^2, -4AC(1/C^2-1)\}$ does not hold for $0 < p_1 < 1$.

3($c$). The inequality $|C|(|B|+4|A|)-|AB| \le 0$ is equivalent to the inequality $44p_1^4 - 68p_1^2 - 16 - p_1^4 \ge 0$, which is false for $p_1 \in (0,1)$.

3($d$). The inequality

$$|AB|-|C|(|B|-4|A|) = \frac{96p_1^4+88p_1^2-15}{1-p1^2} \le 0$$

is equivalent to $96p_1^4+88p_1^2-15 \le 0$, which is true for

$$0 < p_1 \le p_1'' = \frac{1}{2}\sqrt{\frac{\sqrt{211}-11}{6}} \approx 0.38328.$$

From (3.7) and Lemma 2.2,

$$|H_{2,1}(F_{f^{-1}}/2)| \le \tfrac{1}{3}p_1(1-p_1^2)(-|A|+|B|+|C|) = \tfrac{1}{12}(3+8p_1^2-24p_1^4) = h(p_1), \quad (3.9)$$

where $h(x) = 3+8x^2-24x^4$. Since $h'(x) > 0$ in $0 < x \le p_1''$, we have $h(x) \le h(p_1'')$ for $0 < x \le p_1''$. Therefore,

$$|H_{2,1}(F_{f^{-1}}/2)| \le \tfrac{1}{48}(-58+5\sqrt{211}) \approx 0.304775.$$

3($e$). Furthermore, for $p_1'' < p_1 < 1$, from (3.8) and Lemma 2.2,

$$\begin{aligned} |H_{2,1}(F_{f^{-1}}/2)| &\le \frac{1}{24}p_1(1-p_1^2)(|C|+|A|)\sqrt{1-\frac{B^2}{4AC}} \\ &= \frac{1}{6}(12p_1^4-2p_1^2+3)\sqrt{\frac{16-3p_1^2}{39+13p_1^2}} = k(p_1), \end{aligned}$$

where

$$k(x) = \frac{1}{6}\sqrt{\frac{16-3x^2}{39+13x^2}}(12x^4-2x^2+3).$$

As $k'(x) = 0$ has no solution in $(p_1'', 1)$ and $k'(x) > 0$, the maximum occurs at $x = 1$ and we conclude that

$$|H_{2,1}(F_{f^{-1}}/2)| \le k(1) = \tfrac{13}{12} \quad \text{for } p_1'' < x < 1.$$

The desired inequality (3.6) follows from Cases 1–3. For the equality, consider the Koebe function

$$k(z) = \frac{z}{(1-z)^2}.$$

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Clearly, $k \in \mathcal{S}^*$ and it is easy to show that

$$|H_{2,1}(F_{k^{-1}}/2)| = \tfrac{13}{12}.$$

This completes the proof. □

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