

PARTIAL AUTOMORPHISMS OF STABLE C*-ALGEBRAS AND HILBERT C*-BIMODULES

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Abstract

Let A be a C*-algebra and \mathbb{K} the C*-algebra of all compact operators on a countably infinite dimensional Hilbert space. In this note, we shall show that there is an isomorphism of a semigroup of equivalence classes of certain partial automorphisms of $A \otimes \mathbb{K}$ onto a semigroup of equivalence classes of certain countably generated A - A -Hilbert bimodules.

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0. Introduction

Let A be a C*-algebra and \mathbb{K} the C*-algebra of all compact operators on a countably infinite dimensional Hilbert space. Brown, Green and Rieffel [4] defined $\text{Pic}(A)$, the Picard group of A as the group of isomorphic classes of A - A -equivalence bimodules and showed that there is a homomorphism of $\text{Aut}(A \otimes \mathbb{K})$, the group of all automorphisms of $A \otimes \mathbb{K}$ to $\text{Pic}(A)$ and that its kernel is $\text{Int}(A \otimes \mathbb{K})$, the normal subgroup of $\text{Aut}(A \otimes \mathbb{K})$ of all generalized inner automorphisms of $A \otimes \mathbb{K}$. If A is σ -unital, the homomorphism of $\text{Aut}(A \otimes \mathbb{K})$ to $\text{Pic}(A)$ is surjective. Hence $\text{Out}(A \otimes \mathbb{K}) (= \text{Aut}(A \otimes \mathbb{K}) / \text{Int}(A \otimes \mathbb{K}))$ is isomorphic to $\text{Pic}(A)$.

In this note, we shall give a similar result to the above one for certain partial automorphisms of $A \otimes \mathbb{K}$ and certain countably generated A - A -Hilbert bimodules.

For a C*-algebra B , let $\text{Aut}(B)$ be the group of all automorphisms of B and $M(B)$ its multiplier algebra.

1. Equivalence relations and a map

Let A and \mathbb{K} be as above. Let $\text{PAut}(A \otimes \mathbb{K})$ be the set of all partial automorphisms of $A \otimes \mathbb{K}$ defined by Exel [6]. We shall give an equivalence relation \sim as follows; for $\Theta_j = (I_j, J_j, \theta_j) \in \text{PAut}(A \otimes \mathbb{K})$ ($j = 1, 2$), $\Theta_1 \sim \Theta_2$ if $I_1 = I_2$, $J_1 = J_2$ and there is a unitary element $w \in M(I_1)$ such that $\theta_2 = \theta_1 \circ \text{Ad}(w)$. We denote by $[\Theta]$ the equivalence class of a partial automorphism $\Theta \in \text{PAut}(A \otimes \mathbb{K})$. We also denote by $\text{PAut}(A \otimes \mathbb{K})/\sim$ the quotient set of $\text{PAut}(A \otimes \mathbb{K})$ by the above equivalence relation. We shall define a product in $\text{PAut}(A \otimes \mathbb{K})/\sim$ as follows. Let $\Theta_j = (I_j, J_j, \theta_j) \in \text{PAut}(A \otimes \mathbb{K})$ ($j = 1, 2$), and let $I_3 = \theta_1^{-1}(J_1 \cap I_2)$, $J_3 = \theta_2(J_1 \cap I_2)$, $\theta_3 = \theta_2 \circ \theta_1$. Finally, let $\Theta_3 = (I_3, J_3, \theta_3) \in \text{PAut}(A \otimes \mathbb{K})$. We define $[\Theta_1][\Theta_2] = [\Theta_3]$. By routine computations, we can see that $[\Theta_3]$ is independent of the choices of representatives of $[\Theta_1]$ and $[\Theta_2]$. By the above product, $\text{PAut}(A \otimes \mathbb{K})/\sim$ is a semigroup. Let $\Theta_e = (A \otimes \mathbb{K}, A \otimes \mathbb{K}, id_{A \otimes \mathbb{K}}) \in \text{PAut}(A \otimes \mathbb{K})$. Then by easy computations $[\Theta_e]$ is the unit element in $\text{PAut}(A \otimes \mathbb{K})/\sim$ and the group of all invertible elements in $\text{PAut}(A \otimes \mathbb{K})/\sim$ is

$$\{[(A \otimes \mathbb{K}, A \otimes \mathbb{K}, \beta)] \mid \beta \in \text{Aut}(A \otimes \mathbb{K})\} \cong \text{Out}(A \otimes \mathbb{K}).$$

We identify it with $\text{Out}(A \otimes \mathbb{K})$ and denote an element $[(A \otimes \mathbb{K}, A \otimes \mathbb{K}, \beta)] \in \text{PAut}(A \otimes \mathbb{K})/\sim$ by $[\beta]$.

Let $\text{HB}(A)$ be the set of all A - A -Hilbert bimodule isomorphic classes of A - A -Hilbert bimodules defined in Brown, Mingo and Shen [5] and Abadie, Eilers and Exel [1]. For any A - A -Hilbert bimodule X , we denote by $[X]$ the A - A -Hilbert bimodule isomorphic class of X . We define a product in $\text{HB}(A)$ as the relative tensor product with respect to A .

In the same way as in Abadie, Eilers and Exel [1, Example 3.2], for any $\Theta = (I, J, \theta) \in \text{PAut}(A \otimes \mathbb{K})$ we define an $A \otimes \mathbb{K}$ - $A \otimes \mathbb{K}$ -Hilbert bimodule X_Θ as follows. Let X_Θ be the vector space I and the obvious left action of $A \otimes \mathbb{K}$ and the obvious left inner product, but we define the right action of $A \otimes \mathbb{K}$ on X_Θ by $x \cdot a = \theta^{-1}(\theta(x)a)$ for any $a \in A \otimes \mathbb{K}$ and $x \in X_\Theta$ and the right $A \otimes \mathbb{K}$ -valued inner product by $\langle x, y \rangle_{A \otimes \mathbb{K}} = \theta(x^*y)$ for any $x, y \in X_\Theta$.

LEMMA 1.1. *Let $\Theta_j = (I_j, J_j, \theta_j) \in \text{PAut}(A \otimes \mathbb{K})$ ($j = 1, 2$). If $\Theta_1 \sim \Theta_2$, then $X_{\Theta_1} \cong X_{\Theta_2}$ as $A \otimes \mathbb{K}$ - $A \otimes \mathbb{K}$ -Hilbert bimodules.*

PROOF. We shall prove this lemma in the same way as in Brown, Green and Rieffel [4, Proposition 3.1]. Since $\Theta_1 \sim \Theta_2$, $I_1 = I_2$ and $J_1 = J_2$. Put $I = I_1 = I_2$ and $J = J_1 = J_2$. Then there is a unitary element $w \in M(I)$ such that $\theta_2 = \theta_1 \circ \text{Ad}(w)$. Let Φ be a map from X_{Θ_1} to X_{Θ_2} defined by $\Phi(x) = xw$ for any $x \in X_{\Theta_1}$. Then by direct

computations, we can see that Φ is an $A \otimes \mathbb{K}$ - $A \otimes \mathbb{K}$ -Hilbert bimodule isomorphism of X_{Θ_1} onto X_{Θ_2} since $\theta_2 = \theta_1 \circ \text{Ad}(w)$. Hence we obtain the conclusion. \square

By Lemma 1.1, the map $[\Theta] \in \text{PAut}(A \otimes \mathbb{K})/\sim \rightarrow [X_\Theta] \in \text{HB}(A \otimes \mathbb{K})$ can be defined.

LEMMA 1.2. *The map $[\Theta] \in \text{PAut}(A \otimes \mathbb{K})/\sim \rightarrow [X_\Theta] \in \text{HB}(A \otimes \mathbb{K})$ is a semigroup homomorphism.*

PROOF. Let $\Theta_j = (I_j, J_j, \theta_j) \in \text{PAut}(A \otimes \mathbb{K})$ ($j = 1, 2$). Let $\Theta_3 = (\theta_1^{-1}(J_1 \cap I_2), \theta_2(J_1 \cap I_2), \theta_2 \circ \theta_1) \in \text{PAut}(A \otimes \mathbb{K})$. Then by the definition of the product in $\text{PAut}(A \otimes \mathbb{K})/\sim$, $[\Theta_3] = [\Theta_1][\Theta_2]$. Hence it suffices to show that $X_{\Theta_1} \otimes_{A \otimes \mathbb{K}} X_{\Theta_2} \cong X_{\Theta_3}$ as $A \otimes \mathbb{K}$ - $A \otimes \mathbb{K}$ -Hilbert bimodules. We note that $X_{\Theta_1} = I_1$, $X_{\Theta_2} = I_2$ and $X_{\Theta_3} = \theta_1^{-1}(J_1 \cap I_2)$ as vector spaces. Let Φ be a map of $X_{\Theta_1} \otimes_{A \otimes \mathbb{K}} X_{\Theta_2}$ to X_{Θ_3} , defined by $\Phi(x_1 \otimes x_2) = \theta_1^{-1}(\theta_1(x_1)x_2)$ for any $x_1 \in X_{\Theta_1}$ and $x_2 \in X_{\Theta_2}$. For any $a \in A \otimes \mathbb{K}$ and $x_1 \in X_{\Theta_1}, x_2 \in X_{\Theta_2}$ with $x_1, x_2 \geq 0$,

$$\Phi(x_1 \otimes (a \cdot x_2)) = \theta_1^{-1}(\theta_1(x_1)ax_2) = \Phi((x_1 \cdot a) \otimes x_2).$$

Thus Φ is well defined. It is also clear that Φ is surjective. Furthermore, by routine computations, Φ preserves the both inner products. Hence, by the remark after Jensen and Thomsen [7, Definition 1.1.18], $X_{\Theta_1} \otimes_{A \otimes \mathbb{K}} X_{\Theta_2} \cong X_{\Theta_3}$ as $A \otimes \mathbb{K}$ - $A \otimes \mathbb{K}$ -Hilbert bimodules. Therefore, we obtain the conclusion. \square

For any A - A -Hilbert bimodule, let $I_L(X)$ be the closure of linear span of $\{{}_A\langle x, y \mid x, y \in X \rangle$ and $I_R(X)$ the closure of linear span of $\{\langle x, y \rangle_A \mid x, y \in X\}$. By Brown, Mingo and Shen [5, Remark 1.9], $I_L(X)$ and $I_R(X)$ are closed two-sided ideals of A and by restriction we regard X as an $I_L(X)$ - $I_R(X)$ -equivalence bimodule.

LEMMA 1.3. *The homomorphism $[\Theta] \in \text{PAut}(A \otimes \mathbb{K})/\sim \rightarrow [X_\Theta] \in \text{HB}(A \otimes \mathbb{K})$ is injective.*

PROOF. Let $\Theta_j \in \text{PAut}(A \otimes \mathbb{K})$ ($j = 1, 2$). We suppose that $X_{\Theta_1} \cong X_{\Theta_2}$ as $A \otimes \mathbb{K}$ - $A \otimes \mathbb{K}$ -Hilbert bimodules. Thus there is an $A \otimes \mathbb{K}$ - $A \otimes \mathbb{K}$ -Hilbert bimodule isomorphism Φ of X_{Θ_1} onto X_{Θ_2} . Then since Φ preserves the both inner products, $I_L(X_{\Theta_1}) = I_L(X_{\Theta_2})$ and $I_R(X_{\Theta_1}) = I_R(X_{\Theta_2})$. On the other hand, $I_L(X_{\Theta_j}) = I_j$, $I_R(X_{\Theta_j}) = J_j$ by the definition of X_{Θ_j} for $j = 1, 2$. Thus $I_1 = I_2$ and $J_1 = J_2$. We regard X_{Θ_1} and X_{Θ_2} as I - J -equivalence bimodules where $I = I_1 = I_2$ and $J = J_1 = J_2$. Then $X_{\Theta_1} \cong X_{\Theta_2}$ as I - J -equivalence bimodules. Hence, by Brown, Green and Rieffel [4, Corollary 3.2], there is a unitary element $w \in M(I)$ such that $\theta_2 = \theta_1 \circ \text{Ad}(w)$. Therefore, $\Theta_1 \sim \Theta_2$. \square

Let B be a C^* -algebra. For each A - B -Hilbert bimodule X , let \tilde{X} be its dual Hilbert C^* -bimodule. Let \mathbb{H}_A be the standard equivalence bimodule between $A \otimes \mathbb{K}$ and A , that is, $\mathbb{H}_A = (A \otimes \mathbb{K})(1 \otimes e)$, where 1 is the unit element in $M(A)$, e is a rank one projection in \mathbb{K} and we identify A with $A \otimes e$. For any $\Theta \in \text{PAut}(A \otimes \mathbb{K})$, let $Y_\Theta = \mathbb{H}_A \otimes_{A \otimes \mathbb{K}} X_\Theta \otimes_{A \otimes \mathbb{K}} \mathbb{H}_A$. Then the map $[X] \rightarrow [\mathbb{H}_A \otimes_{A \otimes \mathbb{K}} X \otimes_{A \otimes \mathbb{K}} \mathbb{H}_A]$ is an isomorphism of $\text{HB}(A \otimes \mathbb{K})$ onto $\text{HB}(A)$. Thus the map $[\Theta] \in \text{PAut}(A \otimes \mathbb{K})/\sim \rightarrow [Y_\Theta] \in \text{HB}(A)$ is injective.

COROLLARY 1.4. *The map $[\Theta] \in \text{PAut}(A \otimes \mathbb{K})/\sim \rightarrow [Y_\Theta] \in \text{HB}(A)$ is an injective homomorphism.*

2. A certain class of countably generated Hilbert bimodules and a certain class of partial automorphisms

Let Y be an A - A -Hilbert bimodule. We say that Y is countably generated as equivalence bimodules if Y is a countably generated left Hilbert $I_L(Y)$ -module and Y is a countably generated right Hilbert $I_R(Y)$ -module.

Let Y be an A - A -Hilbert bimodule countably generated as equivalence bimodules. By Blackadar [2, Corollary 13.6.3] $I_L(Y)$ and $I_R(Y)$ are σ -unital. Put $I = I_L(Y) \otimes \mathbb{K}$, $J = I_R(Y) \otimes \mathbb{K}$. Then I and J are also σ -unital by Brown, Green and Rieffel [4, Proposition 2.4]. Let $X = \mathbb{H}_A \otimes_A Y \otimes_A \tilde{\mathbb{H}}_A$. Then

$$\begin{aligned} I_L(X) &= \overline{{}_{A \otimes \mathbb{K}} \langle \mathbb{H}_A \otimes_A \langle Y \otimes_A \langle \mathbb{H}_A, \tilde{\mathbb{H}}_A \rangle, Y \rangle, \mathbb{H}_A \rangle} \\ &= \overline{{}_{A \otimes \mathbb{K}} \langle \mathbb{H}_A \otimes_A \langle Y \otimes_A Y \rangle, \mathbb{H}_A \rangle} = \overline{{}_{A \otimes \mathbb{K}} \langle \mathbb{H}_A \otimes_A I_L(Y), \mathbb{H}_A \rangle} = I_L(Y) \otimes \mathbb{K} \end{aligned}$$

since $\overline{Y \otimes_A Y} = Y$ by Brown, Mingo and Shen [5, Proposition 1.7]. Similarly $I_R(X) = I_R(Y) \otimes \mathbb{K}$.

LEMMA 2.1. *Let Y be an A - A -Hilbert bimodule countably generated as equivalence bimodules. Then there is a partial automorphism Θ_Y of $A \otimes \mathbb{K}$ such that $Y_{\Theta_Y} \cong Y$ as A - A -Hilbert bimodules.*

PROOF. Let I and J be as above. Then I and J are σ -unital. Hence, by Brown, Green and Rieffel [4, Theorem 3.4], there is an isomorphism θ of I onto J such that $X_\theta \cong X$ as I - J -equivalence bimodules where X_θ is an I - J -equivalence bimodule induced by the isomorphism θ of I onto J which is defined in Brown, Green and Rieffel [4, Section 3] and $X = \mathbb{H}_A \otimes_A Y \otimes_A \tilde{\mathbb{H}}_A$. Let Θ_Y be a partial automorphism of $A \otimes \mathbb{K}$ defined by $\Theta_Y = (I, J, \theta)$. Then by the definition of X_{Θ_Y} , $X_{\Theta_Y} = X_\theta$ as I - J -equivalence bimodules. Hence $X_{\Theta_Y} \cong X$ as I - J -equivalence bimodules. By the remark after Jensen and Thomsen [7, Definition 1.1.18], $X_{\Theta_Y} \cong X$ as $A \otimes \mathbb{K}$ - $A \otimes \mathbb{K}$ -Hilbert bimodules. Therefore, $Y_{\Theta_Y} \cong Y$ as A - A -Hilbert bimodules. \square

Let $\text{PAut}_\sigma(A \otimes \mathbb{K})$ be the set of all partial automorphisms $\Theta = (I, J, \theta)$ of $A \otimes \mathbb{K}$ satisfying that I and J are σ -unital. Let $\text{HB}_c(A)$ be the subsemigroup of all A - A -Hilbert bimodules countably generated as equivalence bimodules.

LEMMA 2.2. $\text{HB}_c(A)$ is the image of a subsemigroup $\text{PAut}_\sigma(A \otimes \mathbb{K})/\sim$ by the homomorphism $[\Theta] \in \text{PAut}(A \otimes \mathbb{K})/\sim \rightarrow [Y_\Theta] \in \text{HB}(A)$.

PROOF. By the proof of Lemma 2.1, we see that $\text{HB}_c(A)$ is contained in the image of $\text{PAut}_\sigma(A \otimes \mathbb{K})/\sim$ by the map $[\Theta] \in \text{PAut}(A \otimes \mathbb{K})/\sim \rightarrow [Y_\Theta] \in \text{HB}(A)$. Hence we have only to show the inverse inclusion. Let $\Theta = (I, J, \theta) \in \text{PAut}_\sigma(A \otimes \mathbb{K})$. Then since $\overline{X_\Theta(A \otimes \mathbb{K})} = X_\Theta$ by Brown, Mingo and Shen [5, Proposition 1.7],

$$\begin{aligned} I_L(Y_\Theta) &= \overline{{}_A\langle \tilde{\mathbb{H}}_A \text{ }_{A \otimes \mathbb{K}}\langle X_\Theta \text{ }_{A \otimes \mathbb{K}}\langle \mathbb{H}_A, \mathbb{H}_A \rangle, X_\Theta \rangle, \tilde{\mathbb{H}}_A \rangle} \\ &= \overline{{}_A\langle \tilde{\mathbb{H}}_A \text{ }_{A \otimes \mathbb{K}}\langle X_\Theta, X_\Theta \rangle, \tilde{\mathbb{H}}_A \rangle} = \overline{\langle I\mathbb{H}_A, \mathbb{H}_A \rangle_A} \\ &= (1 \otimes e)I(1 \otimes e). \end{aligned}$$

Let $(A \otimes \mathbb{K})''$ and I'' be the universal von Neumann algebras of $A \otimes \mathbb{K}$ and I , respectively. Then by Pedersen [8], there is a central open projection $p \in (A \otimes \mathbb{K})''$ with $I'' = p(A \otimes \mathbb{K})''$ and $I = p(A \otimes \mathbb{K})'' \cap (A \otimes \mathbb{K})$. Put $q = (1 \otimes e)p$. Then q is a full projection in $M(I)$ and $I_L(Y_\Theta) = qIq$. Since I is σ -unital, by Brown [3, Corollary 2.6], $I_L(Y_\Theta)$ is stably isomorphic to I . Hence, by Brown, Green and Rieffel [4, Proposition 2.4] $I_L(Y_\Theta)$ is σ -unital since I is σ -unital. Similarly, $I_R(Y_\Theta)$ is also σ -unital. Thus, by Blackadar [2, Corollary 13.6.3], Y_Θ is countably generated as equivalence bimodules. Therefore, $[Y_\Theta] \in \text{HB}_c(A)$. □

By Corollary 1.4 and Lemma 2.2, we obtain the following theorem.

THEOREM 2.3. *The map $[\Theta] \rightarrow [Y_\Theta]$ is an isomorphism of $\text{PAut}_\sigma(A \otimes \mathbb{K})/\sim$ onto $\text{HB}_c(A)$.*

Using the proof of Lemma 2.1, we obtain the following corollary.

COROLLARY 2.4. *For any σ -unital closed two-sided ideal I of $A \otimes \mathbb{K}$, there is a σ -unital closed two-sided ideal I_0 of A such that $I \cong I_0 \otimes \mathbb{K}$.*

3. Conjugacy classes

For a C*-algebra A , let $\text{HB}(A)^{-1}$ be the set of all invertible elements in $\text{HB}(A)$. Then $\text{Pic}(A) = \text{HB}(A)^{-1}$. We consider the set of conjugacy classes of elements in $\text{HB}(A)$ by an invertible element in $\text{HB}(A)$. That is, for any $[X_1], [X_2] \in \text{HB}(A)$, $[X_1]$

is equivalent to $[X_2]$, written $[X_1] \sim [X_2]$, if there is an element $[Y] \in \text{Pic}(A)$ such that $[X_2] = [Y][X_1][Y]^{-1}$. We denote by $\text{HB}(A)/\sim$ the quotient set by the above equivalence relation and denote by $[[X]]$ the equivalence class of $[X] \in \text{HB}(A)$. By Brown, Green and Rieffel [4, Proposition 3.1], we can easily consider the following conjecture:

CONJECTURE. *There is a map from $\text{PAut}(A \otimes \mathbb{K})/\approx$ to $\text{HB}(A)/\sim$ induced by the homomorphism of $\text{PAut}(A \otimes \mathbb{K})/\sim$ to $\text{HB}(A)$ defined in Section 2 if we give an appropriate equivalence relation \sim in $\text{PAut}(A \otimes \mathbb{K})/\sim$, where $\text{PAut}(A \otimes \mathbb{K})/\approx$ is the quotient set of $\text{PAut}(A \otimes \mathbb{K})/\sim$ by the appropriate equivalence relation. Furthermore, the map from $\text{PAut}(A \otimes \mathbb{K})/\approx$ to $\text{HB}(A)/\sim$ is injective if A is σ -unital.*

In this section, we shall show this conjecture and obtain the main theorem. First, we shall give an equivalence relation \sim in $\text{PAut}(A \otimes \mathbb{K})/\sim$ as follows; for $\Theta_j = (I_j, J_j, \theta_j) \in \text{PAut}(A \otimes \mathbb{K})$ ($j = 1, 2$), $[\Theta_1] \sim [\Theta_2]$ if there is a $\beta \in \text{Aut}(A \otimes \mathbb{K})$ such that $I_2 = \beta(I_1)$, $J_2 = \beta(J_1)$ and $[\Theta_1] = [(I_1, J_1, \beta^{-1} \circ \theta_2 \circ \beta)]$. We denote by $[[\Theta]]$ the equivalence class of $[\Theta] \in \text{PAut}(A \otimes \mathbb{K})/\sim$.

LEMMA 3.1. *Let $\Theta_j = (I_j, J_j, \theta_j) \in \text{PAut}(A \otimes \mathbb{K})$ for $j = 1, 2$. If $[\Theta_1] \sim [\Theta_2]$, $[Y_{\Theta_1}] \sim [Y_{\Theta_2}]$.*

PROOF. Since $[\Theta_1] \sim [\Theta_2]$, there is a $\beta \in \text{Aut}(A \otimes \mathbb{K})$ such that $I_2 = \beta(I_1)$, $J_2 = \beta(J_1)$ and $[\Theta_1] = [(I_1, J_1, \beta^{-1} \circ \theta_2 \circ \beta)]$. By the definition of the product in $\text{PAut}(A \otimes \mathbb{K})/\sim$,

$$\begin{aligned} [\Theta_1] &= [(I_1, J_1, \beta^{-1} \circ \theta_2 \circ \beta)] \\ &= [(A \otimes \mathbb{K}, A \otimes \mathbb{K}, \beta)][[\Theta_2]][(A \otimes \mathbb{K}, A \otimes \mathbb{K}, \beta^{-1})] = [\beta][[\Theta_2]][\beta]^{-1}. \end{aligned}$$

Thus using the homomorphism of $\text{PAut}(A \otimes \mathbb{K})/\sim$ to $\text{HB}(A)$ defined in Section 2, we see that $[Y_{\Theta_1}] = [Y_\beta][Y_{\Theta_2}][Y_\beta]^{-1}$, where $Y_\beta = \tilde{\mathbb{H}}_A \otimes_{A \otimes \mathbb{K}} X_\beta \otimes_{A \otimes \mathbb{K}} \mathbb{H}_A$ and X_β is an $A \otimes \mathbb{K}$ - $A \otimes \mathbb{K}$ -equivalence bimodule induced by β , which is defined in Brown, Green and Rieffel [4, Section 3]. Therefore, we obtain the conclusion. \square

By Lemma 3.1, we can define a map

$$(1) \quad [[\Theta]] \in \text{PAut}(A \otimes \mathbb{K})/\approx \rightarrow [[Y_\Theta]] \in \text{HB}(A)/\sim.$$

LEMMA 3.2. *If A is a σ -unital C^* -algebra, the map (1) is injective.*

PROOF. Let $\Theta_j = (I_j, J_j, \theta_j) \in \text{PAut}(A \otimes \mathbb{K})$ ($j = 1, 2$) with $[Y_{\Theta_1}] \sim [Y_{\Theta_2}]$. Since A is σ -unital, by Brown, Green and Rieffel [4, Corollary 3.5], there is a $\beta \in \text{Aut}(A \otimes \mathbb{K})$ such that $[Y_{\Theta_1}] = [Y_\beta][Y_{\Theta_2}][Y_\beta]^{-1}$ where Y_β is an A - A -equivalence bimodule defined in the proof of Lemma 3.1. Since the homomorphism $[\Theta] \in \text{PAut}(A \otimes \mathbb{K})/\sim \rightarrow [Y_\Theta] \in \text{HB}(A)$ is injective, we obtain that $[\Theta_1] = [\beta][[\Theta_2]][\beta]^{-1}$. \square

THEOREM 3.3. *Let A be a σ -unital C*-algebra. Then the map*

$$[[\Theta]] \in \text{PAut}_\sigma(A \otimes \mathbb{K})/\approx \rightarrow [[Y_\Theta]] \in \text{HB}_c(A)/\sim$$

is a bijection.

PROOF. The result follows immediately from Theorem 2.3 and Lemma 3.2. □

4. Crossed products

In this section, we shall consider a crossed product of a C*-algebra by a Hilbert C*-bimodule defined in Abadie, Eilers and Exel [1].

Let A be a C*-algebra and Y an A - A -Hilbert bimodule countably generated as equivalence bimodules. Then by Abadie, Eilers and Exel [1, Remark 3.4],

$$(A \times_Y \mathbb{Z}) \otimes \mathbb{K} \cong (A \otimes \mathbb{K}) \times_{\Theta_Y} \mathbb{Z},$$

where $(A \otimes \mathbb{K}) \times_{\Theta_Y} \mathbb{Z}$ is the crossed product of $A \otimes \mathbb{K}$ by Θ_Y which is defined in Exel [6] and Θ_Y is a partial automorphism of $A \otimes \mathbb{K}$ induced by Y which is defined in Section 2. Hence we obtain the following proposition.

PROPOSITION 4.1. *Let A be a C*-algebra and Y an A - A -Hilbert bimodule countably generated as equivalence bimodules. Then*

$$A \times_Y \mathbb{Z} \cong (1 \otimes e)((A \otimes \mathbb{K}) \times_{\Theta_Y} \mathbb{Z})(1 \otimes e),$$

where Θ_Y is the partial automorphism of $A \otimes \mathbb{K}$ induced by Y and 1 is the unit element in $M(A)$, e is a rank one projection in \mathbb{K} . In particular, if A is a simple σ -unital C*-algebra, there is a $\beta \in \text{Aut}(A \otimes \mathbb{K})$ which is uniquely determined up to multiplication by a generalized inner automorphism of $A \otimes \mathbb{K}$ such that

$$A \times_Y \mathbb{Z} \cong (1 \otimes e)((A \otimes \mathbb{K}) \times_\beta \mathbb{Z})(1 \otimes e).$$

COROLLARY 4.2. *Let A be a simple σ -unital C*-algebra and Y an A - A -equivalence bimodule. Then $A \times_Y \mathbb{Z}$ is simple if and only if for any $n \in \mathbb{Z} \setminus \{0\}$, $[Y]^n \neq [A]$ in $\text{Pic}(A)$.*

PROOF. Let β be an automorphism of $A \otimes \mathbb{K}$ such that

$$A \times_Y \mathbb{Z} \cong (1 \otimes e)((A \otimes \mathbb{K}) \times_\beta \mathbb{Z})(1 \otimes e).$$

Hence $A \times_\beta \mathbb{Z}$ is simple if and only if $\beta^n \notin \text{Int}(A \otimes \mathbb{K})$ for any $n \in \mathbb{Z} \setminus \{0\}$ by Pedersen [8, Corollary 8.9.10]. By the proof of Lemma 2.1, $Y \cong \mathbb{H}_A \otimes_{A \otimes \mathbb{K}} X_\beta \otimes_{A \otimes \mathbb{K}} \mathbb{H}_A$ where X_β is the $A \otimes \mathbb{K}$ - $A \otimes \mathbb{K}$ -equivalence bimodule induced by β . Therefore, by Brown, Green and Rieffel [4, Corollary 3.5], we obtain the conclusion. □

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