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CONJECTURE OF D. R. HUGHES EXTENDED TO GENERALIZED ANDRÉ PLANES

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1. Introduction. In 1967 Foulser [**1**] defined a class of translation planes, called generalized André planes or λ-planes and discussed the associated autotopism collineation groups. While discussing these collineation groups he raised the following question:

> "Are there collineations of a λ plane which move the axes but do not interchange them?".

In this context, Foulser mentioned a conjecture of D. R. Hughes that among the André planes, only the Hall planes have collineations moving the axes without interchanging them. Wilke [**4**] answered Foulser's question partially by showing that the conjecture of Hughes is indeed correct. Recently, Foulser [**2**] has shown that possibly with a certain exception the Hall planes are the only generalized André planes which have collineations moving the axes without interchanging them. Our aim in this paper is to give an alternate proof, which is completely general, and is in the style of the original problem.

2. Throughout this paper Foulser's notation [**1**] is used. In what follows let $F(+, \circ)$ be a proper λ-system defined from the field $\mathrm{GF}(q^d)$ with $q = p^s$, p a prime, d and s natural numbers. Let λ be the mapping used to define $F(+, \circ)$ from $\mathrm{GF}(q^d)$ and let Π be the projective plane coordinatized by $F(+, \circ)$ using Hall's method [**3**, p. 353]. Since the only proper Veblen-Wedderburn system of order 9 is a Hall system, we consider the proper λ-systems of order $n \neq 9$.

The proofs of the following theorems may be found in the references indicated.

THEOREM 2.1 [**1**, Lemma 6.1]. *No collineation of* Π *moves one of* X *and* Y *and fixes the other.*

THEOREM 2.2 [**4**, Theorem 8]. *If* ν *is a collineation of* Π *such that* $Y\nu = (r)$, $X\nu = (s) \neq (0)$, *then there is a collineation* δ *such that* $Y\delta = (r_1)$, $X\delta = (s_1) \neq (0)$, *and* $r_1 + s_1 \neq 0$.

THEOREM 2.3 [**4**, Theorem 9]. *Let* ν *be a collineation of* Π *such that* $Y\nu = (r)$, $X\nu = (s) \neq (0)$, *and* $r + s \neq 0$. *If* δ *is an* $(r) - O(s)$ *perspectivity of* Π *with* $Y\delta = (a) \neq (0)$, *and* $X\delta = (b)$, *then*

$$\lambda(r) = \lambda(s) = \lambda(a) = \lambda(b) \qquad \text{and} \qquad a \cdot b + r \cdot s = b \cdot (r + s),$$

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where "·" is the multiplication in the field from which the λ-system $F(+, \circ)$ is defined.

THEOREM 2.4 (Albert and Hughes [**1**, Theorem 7.6]). *Let $F(+,*)$ be a λ-system defined over* $\mathrm{GF}(q^2)$, *q a prime power, where $\lambda(i) = 0$ if $i \equiv 0 \pmod{q-1}$ and $\lambda(i) = 1$ if $i \not\equiv 0 \pmod{q-1}$. Then $F(+,*)$ coordinatizes a Hall plane.*

THEOREM 2.5 [**4**, Theorems 15 and 16]. *Let $F(+, \square)$ be a finite left André system and let Π_1 be the associated projective plane. If there exists a collineation of Π_1 which moves the two axes of Π_1 without interchanging them, then $F(+, \square)$ is isotopic to $F(+,*)$ of Theorem* 2.4 *and consequently coordinatizes a Hall plane.*

3. In this section we write ab in place of $a \cdot b$. Let T be the automorphism of $\mathrm{GF}(q^d)$ defined by $xT = x^q$. Throughout this section T^{-k} denotes $(T^{-1})^k$ and x/y denotes xy^{-1}, where $T(T^{-1})$ is the identity mapping and $yy^{-1} = 1$.

THEOREM 3.1. *The point (u, v) is mapped into (u', v') by the $(r) - O(s)$ perspectivity δ of Theorem* 2.3, *where*

$$u' = (((r + s - a)T^{-\lambda(r)})u - (v)T^{-\lambda(r)})/(r - a)T^{-\lambda(r)} \tag{3.1}$$

and

$$v' = (rs(uT^{\lambda(r)}) - av)/(r - a). \tag{3.2}$$

Proof. Let A be the point (u, v) and let B be the point where the line through A and (r) meets the line through O and (s). Let C be the point where the line through A and Y meets the line through O and (s). Let D be the point where the line through (a) and C meets the line through (r) and B. From the properties of the $(r) - O(s)$ perspectivity δ, it follows that D is the image of A under δ. The equation of the line through (r) and A is

$$y = r \circ x - r \circ u + v. \tag{3.3}$$

Since the point C lies on the lines $y = s \circ x$ and $x = u$, we see that C has coordinates $(u, s \circ u)$. The line through (a) and C has the equation

$$y = a \circ x - a \circ u + s \circ u. \tag{3.4}$$

From (3.3) and (3.4) it follows that

$$r \circ u' - a \circ u' = r \circ u + s \circ u - a \circ u - v. \tag{3.5}$$

Theorem 3.1 then follows from the fact that

$$\lambda(r) = \lambda(s) = \lambda(a) = \lambda(b).$$

THEOREM 3.2. *If $(a')\delta = Y$ and $(b')\delta = X$, then $a' = r + s - a$, $b' = rs/a$, and $b = rs/a'$.*

Proof. Let $A = (1, a')$ be a point on the line through (a') and O. Let B be the point where the line through A and (r) meets the line through Y and O.

Obviously $A\delta = B$, where B has coordinates $(0, -r + a')$. Let C be the point where the line through A and Y meets the line through O and (s). The points (a), B, and $(1, s)$ are on the same line. It therefore follows that $r + s - a = a'$. The rest of the theorem follows from Theorem 2.3.

THEOREM 3.3. *Let* $(m)\delta = (M)$, *where* $(m) \neq (a'), (b')$. *Then* $h(x)$ *is the zero polynomial, where* $h(x) = l_1(xT^{\lambda(M)}) - l_2(xT^{z}) - l_3(xT^{\lambda(r)}) + l_4(xT^{\lambda(m)})$, *where*

$$\begin{aligned} l_1 &= ((a')T^{-\lambda(r)+\lambda(M)})/((a' - m)T^{\lambda(M)-\lambda(r)}),\\ l_2 &= (mT^{\lambda(M)-\lambda(r)})/((a' - m)T^{\lambda(M)-\lambda(r)}),\\ l_3 &= rs/(rs - am),\\ l_4 &= am/(rs - am), \end{aligned}$$

with $\lambda(M) + \lambda(m) - \lambda(r) \equiv z \pmod{d}$.

Proof. The points $A\,(x, m(xT^{\lambda(m)}))$ lie on the line $[m, 0]$ and by Theorem 3.1 the points A are mapped onto the points $B(u, v)$, where u and v are functions of x and m. It then follows that the points $B(u, v)$ lie on the line $[M, 0]$, and therefore $v = M \circ u$. Let A_1 be the point A for $x = 1$ and $B_1 = (u_1, v_1)$. Then $M \circ u_1 = v_1$. It then follows that

$$M(u_1T^{\lambda(M)}) = v_1 \qquad \text{and} \qquad M(u_xT^{\lambda(M)}) = v_x$$

where $B_x(u_x, v_x)$ is the image of $A_x(x, m(xT^{\lambda(m)}))$. Eliminating M, we obtain

$$(u_x/u_1)T^{\lambda(M)} = v_x/v_1. \tag{3.6}$$

Simplification of (3.6) then leads to the relation $h(x) = 0$ for all $x \in \mathrm{GF}(q^d)$. Since the degree of $h(x)$ is less than q^d and $h(x)$ vanishes for all $x \in \mathrm{GF}(q^d)$, we see that $h(x)$ is the zero polynomial.

THEOREM 3.4. *Let* $A = \{\lambda(m) \in L_\infty \mid \lambda(m) = \lambda(r)\} \cup \{X, Y\}$. *The collineation* δ *permutes the points of* A *among themselves.*

Proof. If $\lambda(r) = \lambda(m)$, then $h(x) = (xT^{\lambda(M)}) - (xT^{\lambda(r)}) = 0$ for all x implying that $\lambda(M) = \lambda(r)$. Hence if $(m) \neq (a'), (b'), (m)\delta = (M)$, and $\lambda(m) = \lambda(r)$, then $\lambda(M) = \lambda(r)$ and the theorem follows.

THEOREM 3.5. *If there is a collineation* γ *of* Π *such that* $Y\gamma = (r) \neq (0), (\infty)$, *then* d *is even and* λ *has only two values,* 0 *and* $d/2$.

Proof. In view of Theorem 2.2 we may suppose that $r + s \neq 0$ where $X\gamma = (s)$. There is a one-to-one correspondence between $Y - OX$ perspectivities and the elements of N_l, the left nucleus of $F'(F' = \{x \in F, x \neq 0\})$. Further, the group N_l does not consist of the identity alone. Let α be a non-identity $Y - OX$ perspectivity. Let $\delta = \gamma^{-1}\alpha\gamma$. Then δ is a non-identity $(r) - O(s)$ perspectivity. Theorems 2.1–2.3, 3.1–3.4 imply that $h(x)$ of Theorem 3.3 is the zero polynomial. Suppose that $\lambda(M) = \lambda(r)$. It follows from Theorem 3.4 that $\lambda(M) = \lambda(m) = \lambda(r)$. Then in order that $h(x)$ be the zero polynomial, it is necessary that $z \equiv \lambda(m) \pmod d$, since $l_2 \neq 0$. Suppose that $\lambda(M) \neq \lambda(r)$. As before, a necessary condition for $h(x)$ to be the zero polynomial is that $\lambda(M) = \lambda(m)$ and $\lambda(M) + \lambda(m) - \lambda(r) \equiv \lambda(r) \pmod d$. These necessary conditions may be restated as follows.

Case 1. If $\lambda(M) = \lambda(r)$, then $\lambda(M) + \lambda(m) - \lambda(r) \equiv \lambda(m) \pmod d$,

Case 2. If $\lambda(M) \neq \lambda(r)$, then
(i) $\lambda(M) = \lambda(m)$ and
(ii) $2\lambda(m) \equiv 2\lambda(r) \pmod d$.

It may be noted that in either case, $\lambda(M) = \lambda(m)$.

Let $0 \neq t \in \mathrm{GF}(q^d)$ be such that $\lambda(t) = 0$, if $\lambda(r) \neq 0$ and $\lambda(t) \neq 0$, if $\lambda(r) = 0$. Such a t exists since we are dealing with a proper λ-system. The following two congruences now follow from Case 2.

$$\begin{aligned} 2\lambda(r) &\equiv 0 \pmod d \qquad \text{if } \lambda(r) \neq 0, \\ 2\lambda(t) &\equiv 0 \pmod d \qquad \text{if } \lambda(r) = 0. \end{aligned}$$

It may be concluded from the above congruences that d is an even integer and that $\lambda(r) = d/2$ if $\lambda(r) \neq 0$.

Suppose that $0 \neq w \in \mathrm{GF}(q^d)$ and $\lambda(w) \neq d/2$. Since $\lambda(r) = 0$ or $d/2$, Case 2 reduces to $2\lambda(w) \equiv 0 \pmod d$ implying that $\lambda(w) = 0$. Thus d is an even integer and λ has only two values 0 and $d/2$.

THEOREM 3.6. *The André planes which are the Hall planes are the only proper λ-planes which have collineations moving $(\infty) \to (m)$, where $m \neq 0, \infty$.*

Proof. Suppose that γ is a collineation of Π such that $Y\gamma = (r) \neq (0), (\infty)$ and $X\gamma = (s)$. In view of Theorem 2.2 we may take $r + s \neq 0$. From Theorem 3.5 we see that d is an even integer having only two values, 0 and $d/2$. It then follows that $F(+, \circ)$ is a λ-system of order q^d with kern $K = \mathrm{GF}(q^{d/2})$, and hence is an André system [**1**, § 7]. The theorem now follows from Theorems 2.4 and 2.5.

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