THE SOLUBLE SUBGROUPS OF A ONE-RELATOR GROUP WITH TORSION

Dedicated to the memory of Hanna Neumann

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1. Introduction

In a survey article [1] Baumslag posed the problem of determining the abelian subgroups of a one-relator group. The solution of this problem was stated but not proved in [5], and partly solved by Moldavanskii [4]. In this paper it will be proved that the centralizer of every non-trivial element in a one-relator group with torsion is cyclic, and that the soluble subgroups of a one-relator group with torsion are cyclic groups or the infinite dihedral group. That both types of groups may occur as subgroups is easily seen by considering

$$G = gp(a, b \mid a^2).$$

This is a one-relator group with torsion with both finite and infinite cyclic subgroups, and the subgroup generated by a and $b^{-1}ab$ is the free product of two cyclic groups of order 2, which is isomorphic to the infinite dihedral group

$$gp(x, y \mid x^2, x^{-1}yxy).$$

The proof follows the general argument of the proof of the Freiheitssatz [3]. In that proof the main tool is the free product with amalgamation, which leads one to ask what soluble subgroups can occur in a free product with amalgamation. Without some conditions on the amalgamated subgroup, one can say little, but if the amalgamated subgroup satisfies certain restrictive conditions, one can keep track of the soluble subgroups. The appropriate condition for the purposes of this paper is that of malnormality.

2. Malnormal subgroups

Let H be a subgroup of a group G. Then H is a malnormal subgroup of G if for all $g \in G$

$$g^{-1}Hg \cap H \neq 1 \Rightarrow g \in H.$$

It is clear from the definition that no element outside a malnormal subgroup H can commute with a non-trivial element of H. In particular no non-trivial element of H can have a root outside H. Another property easily derived from the definition is that malnormality is transitive. Further facts concerning malnormal subgroups are given in the following lemmas.

LEMMA 2.1 *Let* $G = gp(a, b, \cdots, c, t \mid R^n)\, n > 1$ *where* R *is a cyclically reduced word in the generators* $a, b, \cdots, c, t$. *Then every subset of these generators of* G *generates a malnormal subgroup of* G. *Moreover, suppose* $w_1(a, b, \cdots, c)$ *is a word which when cyclically reduced as a word in the free group generated by* $a, b, \cdots, c, t$ *involves the generator* a *non-trivially, and* $w_2(b, \cdots, c, t)$ *is a word which when cyclically reduced involves* t *non-trivially. Then* w_1 *and* w_2 *are not conjugates in* G *if* R *involves* a, t *non-trivially.*

The proof of this lemma follows the usual induction argument and is to be published in another paper. This is a useful lemma, and provides the key to proving one-relator groups with torsion have a soluble conjugacy problem.

LEMMA 2.2. *Let* C *be the free product of groups* A *and* B *with a subgroup* J *amalgamated. If* J *is a malnormal subgroup of the factors* A *and* B *then* A *and* B *are malnormal subgroups of* C.

PROOF. From the symmetry between A and B in C it will suffice to prove that A is a malnormal subgroup of C. Suppose

$$g^{-1}a_1 g = a_2, g \in C, 1 \neq a_1, a_2 \in A.$$

Let left coset representatives of J in A and B be chosen, taking 1 as the coset representative for J, and suppose the normal forms for g, a_1, a_2 are

$$g = s_1 s_2 \cdots s_m \hat{j}, \quad a_1 = t_1 j_1, \quad a_2 = t_2 j_2$$

where $t_1, t_2 \in A, \hat{j}, j_1, j_2 \in J$ and $s_1, s_2, \cdots, s_m$ are coset representatives alternately from A and B. It is required to prove that $g \in A$. This will be done by induction on $|g|$, the length of g in normal form. If $|g| = 0$, then $g \in J \subset A$. Suppose that for all elements $h \in C$ with $|h| < |g|$, it has been shown that for all non-trivial elements a, a' of $A, h^{-1}ah = a'$ implies $h \in A$. Then

$$t_1 j_1 s_1 s_2 \cdots s_m \hat{j} = a_1 g = g a_2 = s_1 s_2 \cdots s_m \hat{j} t_2 j_2.$$

If $t_1 = 1$ then

$$(s_1^{-1} j_1 s_1) s_2 \cdots s_m \hat{j} = s_2 \cdots s_m \hat{j} t_2 j_2,$$

and by examining the first factor on both sides, the malnormality of J in A implies $m = 0$ and so $g \in A$. If $t_1 \neq 1$ then

$$(s_1^{-1}t_1j_1s_1)s_2\cdots s_m\hat{j} = s_2\cdots s_m\hat{j}t_2j_2.$$

From a consideration of the lengths of normal forms in this equation, s_1 and t_1 belong to the same factor, and $s_1^{-1}t_1j_1s_1 = a_3$ for some element $a_3 \in A$. Hence $a_3s_2\cdots s_m\hat{j} = s_2\cdots s_m\hat{j}t_2j_2$, and by the induction hypothesis $s_2\cdots s_m\hat{j} \in A$. Hence $s_1s_2\cdots s_m\hat{j} \in A$. This completes the proof of the lemma.

A group is said to have the *cyclic centralizer property* if the centralizer of every non-trivial element of the group is cyclic.

LEMMA 2.3. *Let C be the free product of groups A and B amalgamating a subgroup J which is malnormal in A and B. If the groups A and B have the cyclic centralizer property then*

(i) *C has the cyclic centralizer property*

and (ii) *the soluble subgroups of C are cyclic or the infinite dihedral group or are conjugates of subgroups of A or B.*

PROOF. Part (i) is a special case of Theorem 2 of [2]. To prove part (ii), let P be any non-trivial soluble subgroup of C, of soluble length r. Let $\delta(G)$ denote the commutator subgroup $[G, G]$ and define inductively

$$\delta^i(G) = \delta(\delta^{i-1}(G)) \quad i = 1, 2, 3, \cdots$$

where $\delta^0(G) = G$. Then $\delta^{r-1}(P) \neq 1 = \delta^r(P)$. The proof of part (ii) will be divided into two cases.

CASE 1. Suppose $\delta^{r-1}(P) \cap A \neq 1$. Let $a \in \delta^{r-1}(P) \cap A$, $a \neq 1$. Then every element of $\delta^{r-1}(P)$ commutes with the element a and so $\delta^{r-1}(P)$ lies in A, since A is malnormal in C. Since $\delta^{r-1}(P)$ is a normal subgroup of P, malnormality of A in C further implies $P \subset A$. Similarly P is conjugate to a subgroup of A or B if $\delta^{r-1}(P)$ has non-trivial intersection with a conjugate of A or B.

CASE 2. Suppose $\delta^{r-1}(P)$ has trivial intersection with all conjugates of A and B. Since $\delta^{r-1}(P)$ is abelian, it is cyclic. Let u be a generator of $\delta^{r-1}(P)$ which is, without loss of generality, cyclically reduced and of length at least 2. Consequently $\delta^{r-1}(P)$ is an infinite cyclic group. From elementary group theory

$$P/Z(\delta^{r-1}(P)) \simeq H,$$

where $Z(\delta^{r-1}(P))$ is the centralizer in P of $\delta^{r-1}(P)$, and H is a subgroup of $\mathrm{Aut}\,(\delta^{r-1}(P))$, the group of automorphisms of $\delta^{r-1}(P)$. Now $Z(\delta^{r-1}(P))$ is the centralizer of u, and from the cyclic centralizer property, is infinite cyclic, generated by x say. Since $\delta^{r-1}(P)$ is infinite cyclic, $\mathrm{Aut}(\delta^{r-1}(P))$ is the cyclic group of order 2. If H is trivial, then $P = Z(\delta^{r-1}(P))$ and so P is infinite cyclic. If P is not cyclic then H is the cyclic group or order 2, and P is generated by x, and an element w, say, with $w^2 = x^v$ for some integer v. If $v \neq 0$, $Z(w^2) \supset gp(x, w)$, and P is cyclic, so assume $v = 0$.

Since x generates an infinite cyclic group, conjugation by w must be the inverting automorphism,

$$w^{-1}xw = x^{-1}.$$

Thus P is a homomorphic image of the group with presentation

$$gp(x, w \mid w^2, w^{-1}xwx).$$

Since every element in such a group has a canonical form x^{α} or wx^{β} for integers α, β, any further relation will imply x has finite order. That is, any proper homomorphic image of the infinite dihedral group is finite. Since P is infinite, it is the infinite dihedral group.

3. The main theorem

THEOREM. *A one-relator group G with torsion has the following property $\mathfrak{P}$: the soluble subgroups of G are cyclic groups or the infinite dihedral group, and the centralizer of every non-trivial element is a cyclic group.*

PROOF. Let $G = gp(a, b, c, d, \cdots \mid R^n)\, n > 1$ where one may assume R is a cyclically reduced word in $a, b, c, d, \cdots$. The theorem will be proved by induction on the length $\lambda(R^n)$ of R^n as a free word in $a, b, c, d, \cdots$. To simplify the notation, the convention will be adopted of deleting from the discussion the dots which refer to generators other than a, b, c, d, As in the proof of the Freiheitssatz, one considers various cases.

CASE 1. Suppose R contains only one generator. Then G is a free product of cyclic groups and so any subgroup is cyclic or the free product of cyclic groups. Also the centralizer of any non-trivial element is cyclic. A free product $A*B$ has a free subgroup of rank 2 unless $A = 1$, or $B = 1$, or A and B are both of order 2. Since a free group of rank 2 is not soluble, the only possible non-cyclic soluble subgroup is Z_2*Z_2 where Z_2 is the cyclic group of order 2. This proves G has property $\mathfrak{P}$ when R contains only one generator.

CASE 2. Suppose R contains more than one generator, say a, b, c, d, and the exponent sum of one of the generators in R is zero. Without loss of generality let the exponent sum of a in R, denoted by $\sigma_a(R)$, be zero. In this case one considers the normal subgroup N of G generated by b, c, d, and shows that N has the required properties. Since N is nicely situated in G, in fact G/N is infinite cyclic, one can then proceed to prove that G has the required properties.To obtain a presentation for N one uses a Reidemeister-Schreier rewriting process, see [3], and takes as generators for N

$$b_i = a^{-i}ba^i, \quad c_i = a^{-i}ca^i, \quad d_i = a^{-i}da^i$$

where i takes all integral values, and the defining relators of N are obtained by rewriting $a^{-i}R^na^i$, where i takes all integral values.

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Let $a^{-i}R^n a^i$ be rewritten in terms of the generators of N. Then it follows easily that the rewritten relator is cyclically reduced and is a power R_i^n where R_i is $a^{-i}Ra^i$ rewritten. Since a occurs in R^n, the rewritten words R_i^n will have shorter length than R^n, that is

$$\lambda(R_i^n) < \lambda(R^n).$$

A presentation for N is

$$N = gp(b_i, c_i, d_i \text{ (for all } i \in I) \mid R_i^n \text{(for all } i \in I))$$

where I denotes the set of integers. As usual in one-relator group theory, N can be constructed from "smaller" one-relator groups using a generalized free product construction. For each integer i define

$$N_i = gp(b_i, \cdots, b_{\mu+i}, c_j, d_j \text{ (all integers } j) \mid R_i^n)$$

where, without loss of generality, zero and μ are taken to be the smallest and largest b-subscript respectively occurring in R_0. Then using the Freiheitssatz, N may be constructed from these one-relator groups N_i as

$$N = \bigcup_{k=0}^{\infty} K_k \qquad \text{where}$$

$$K_0 = N_0$$

$$K_{2k+1} = \{K_{2k} * N_{k+1}; J_{k+1}\} \qquad k \geqq 0$$

$$K_{2k} = \{K_{2k-1} * N_{-k}; J_{-k}\} \qquad k > 0$$

where if $i > 0$

$$J_i = gp(b_i, \cdots, b_{\mu+i-1}, c_j, d_j \text{ (all integers } j))$$

and if $i < 0$

$$J_i = gp(b_{i+1}, \cdots, b_{\mu+i}, c_j, d_j \text{ (all integers } j)).$$

One is able to exploit the induction hypothesis because the building blocks N_i which go into the construction of N are one-relator groups with a relator of shorter length than R^n. Thus from the induction hypothesis all the groups N_i have the property $\mathfrak{P}$. Therefore K_0 has property $\mathfrak{P}$. Since K_1 is a generalized free product of two groups each with property $\mathfrak{P}$, amalgamating a malnormal subgroup (by Lemma 2.1), it follows from Lemma 2.3 that K_1 has property $\mathfrak{P}$.

Now K_2 is a generalized free product of two groups K_1 and N_{-1} each of which has property $\mathfrak{P}$. The amalgamated subgroup in this instance is J_{-1} which is a malnormal subgroup of N_{-1} by Lemma 2.1. But J_{-1} is a malnormal subgroup of N_0 by Lemma 2.1 which in turn is a malnormal subgroup of K_1 by Lemma 2.2. Hence J_{-1} is a malnormal subgroup of K_1 by the transitivity property of malnormal subgroups. Thus K_2 has property $\mathfrak{P}$. One may proceed inductively in this fashion to show that all groups K_m have the required property $\mathfrak{P}$ and that $N_{[(m+1)/2]}$ and $N_{-[m/2]}$ are malnormal in K_m (here [ ] denotes the integral part of). For

$$K_{m+1} = \{K_m * N_{(-1)^m[(m+2)/2]}; J_{(-1)^m[(m+2)/2]}\}$$

and $J_{(-1)^m[(m+2)/2]}$ is malnormal in $N_{(-1)^m[(m+2)/2]}$. Also $J_{(-1)^m[(m+2)/2]}$ is malnormal in $N_{(-1)^m[m/2]}$, which is malnormal in K_m by the induction hypothesis, so $J_{(-1)^m[(m+2)/2]}$ is malnormal in K_m. Since K_m and $N_{(-1)^m[(m+2)/2]}$ have property $\mathfrak{P}$, then K_{m+1} has property $\mathfrak{P}$ and $N_{(-1)^m[(m+2)/2]}, N_{-[m/2]}, N_{[(m+1)/2]}$ are malnormal in K_{m+1}. In particular $N_{[(m+2)/2]}$ and $N_{-[(m+1)/2]}$ are malnormal in K_{m+1}. Since the statement is true for $m = 0, 1$, this proves K_m has property $\mathfrak{P}$ for all positive integers m. It is clear from the construction of K_{m+1}, that K_m is a malnormal subgroup of K_{m+1}. Hence by using transitivity, K_m is a malnormal subgroup of K_i for all integers $i > m$. Thus K_m is a malnormal subgroup of N. Hence the centralizer in N of any non-trivial element of K_m is a subgroup of K_m and so is cyclic. If S is any soluble subgroup of N of length r, and x is any element of $\delta^{r-1}(S)$, then $x \in K_m$ for some integer m. Then every element of $\delta^{r-1}(S)$ must lie in K_m. But since $\delta^{r-1}(S)$ is a normal subgroup of S, then $S \subset K_m$. Thus N has property $\mathfrak{P}$.

To show G has property $\mathfrak{P}$, it is convenient to first show the abelian subgroups of G are cyclic. Let A be an abelian subgroup of G not contained in N. Then $A/A \cap N$ is isomorphic to AN/N which is a subgroup of the infinite cyclic group G/N. Now A, being an abelian infinite cyclic extension of a cyclic group is either cyclic or the direct product of a cyclic group and an infinite cyclic group, the latter infinite cyclic factor not in N. Suppose A is not cyclic. Let x, y be generators of A where $1 \neq y \in N, x = a^t\bar{x}$, t an integer, and $\bar{x} \in N$. Without loss of generality assume t is positive and large. Let y when written as an element of the constructed generating set of N be

$$y_0(b_0, \cdots, b_{\mu+k}, c_i, d_i)$$

and again without loss of generality assume this is the shortest possible word representing the element y, and that $b_0, b_{\mu+k}$ where $\mu + k \geqq 0$ are non-trivial in y. Assume further that A has been chosen with $\mu + k$ minimal. Thus no conjugate of y has smaller $\mu + k$ value than has y_0.

Now

$$x^{-1}y^{-1}xy = 1$$

implies

$$\bar{x}^{-1}a^{-t}y^{-1}a^t\bar{x}y = 1, \quad \text{or}$$

$$\bar{x}^{-1}y_t^{-1}\bar{x}y_0 = 1 \tag{3.1}$$

where $y_t = y_0(b_t, \cdots, b_{t+\mu+k}, c_{i+t}, d_{i+t})$ that is, y_t is the same word y_0 except that the subscripts of letters in y_0 are increased by t. One now proves that equation 3.1 is impossible. Suppose t has been chosen such that $t > \mu + k$ and $t + \mu + k > \mu$ so y_t is not in N_0. Then y_0 and y_t are elements of K_{k+t}, and since K_{k+t} is malnormal in N, then $\bar{x} \in K_{k+t}$. Thus equation 3.1 takes place in K_{k+t}. Let K_{k+t} be written as a free product with amalgamation as follows

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$$K_{k+t} = \{gp(N_{-[(k+t)/2]}, \cdots, N_k) * gp(N_{k+1}, \cdots, N_{k+t}); J_{k+1}\}$$

Now y_0 is an element in the first factor and y_t is in the second factor, hence y_0 and y_t can be conjugates only if y_0 is conjugate to an element of J_{k+1}. By the choice of y this implies that $0 \leqq \mu + k \leqq \mu - 1$, and so $y_0 \in N_0$ and is conjugate in N_0 to an element of J_1. But again this is impossible by Lemma 2.1. Thus equation (3.1) is impossible, so the abelian subgroups of G are cyclic. Thus no element outside N can commute with a non-trivial element inside N. Hence the centralizer in G of every element in N is cyclic. Let $w \in G$ and $w \notin N$, and suppose w commutes with g and h. Then w commutes with $g^{-1}h^{-1}gh$ which belongs to N. From the remark above, $g^{-1}h^{-1}gh = 1$, proving the centralizer of w is abelian and therefore cyclic.

Finally suppose S is a soluble subgroup of G not contained in N. Clearly δS is contained in N, and so is either cyclic or infinite dihedral. If δS is infinite dihedral, then

$$S/Z(\delta^2 S) \simeq H$$

where $Z(\delta^2 S)$, the centralizer of $\delta^2 S$, and H, a group of automorphisms of $\delta^2 S$, are cyclic. Hence δS is abelian, contradicting δS being infinite dihedral. Thus δS is cyclic. Let x be an element of S not in N and y a generator of δS. As δS is cyclic there are only finitely many automorphisms of δS, and hence there exists an integer $r \neq 0$ with

$$x^{-r} y x^{r} = y.$$

Since no non-trivial power of x is in N, it follows that y is the identity, and S is cyclic.

CASE 3. Suppose R contains more than one generator and the exponent sum in R for none of the generators is zero. This case is easily reduced to one of the other cases. Suppose

$$\sigma_a(R) = \alpha, \sigma_b(R) = \beta \qquad \alpha, \beta \neq 0.$$

Consider the new group

$$M = gp(x, y, c, d \,|\, R^n(x^\beta, x^{-\alpha}y, c, d)).$$

This is a one relator group with torsion, and contains G as a subgroup under the isomorphism which maps

$$\begin{aligned} a &\to x^\beta \\ b &\to x^{-\alpha}y \\ c &\to c \\ d &\to d. \end{aligned}$$

However when $R^n(x^\beta, x^{-\alpha}y, c, d)$ is cyclically reduced, it may have greater length than $R^n(a, b, c, d)$. This would be due to extra x-symbols. Since $\sigma_x(R^n) = 0$, one may proceed as in Case 2 above. All the x-symbols are removed in rewriting

$R^n(x^\beta, x^{-\alpha}y, c, d)$ and the corresponding relator R_0^n which arises will have shorter length than the original relator $R^n(a, b, c, d)$, and thus the induction hypothesis will apply. Thus one proceeds as in Case 2 to show that the soluble subgroups of M are either cyclic or infinite dihedral, and the centralizer of any non-trivial element of M is cyclic. Since G is a subgroup of M, the same property holds for G, completing the proof of the theorem.

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