



RESEARCH ARTICLE

Lie algebra actions on module categories for truncated shifted yangians

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Abstract

We develop a theory of parabolic induction and restriction functors relating modules over Coulomb branch algebras, in the sense of Braverman-Finkelberg-Nakajima. Our functors generalize Bezrukavnikov-Etingof’s induction and restriction functors for Cherednik algebras, but their definition uses different tools.

After this general definition, we focus on quiver gauge theories attached to a quiver Γ . The induction and restriction functors allow us to define a categorical action of the corresponding symmetric Kac-Moody algebra \mathfrak{g}_Γ on category \mathcal{O} for these Coulomb branch algebras. When Γ is of Dynkin type, the Coulomb branch algebras are truncated shifted Yangians and quantize generalized affine Grassmannian slices. Thus, we regard our action as a categorification of the geometric Satake correspondence.

To establish this categorical action, we define a new class of ‘flavoured’ KLRW algebras, which are similar to the diagrammatic algebras originally constructed by the second author for the purpose of tensor product categorification. We prove an equivalence between the category of Gelfand-Tsetlin modules over a Coulomb branch algebra and the modules over a flavoured KLRW algebra. This equivalence relates the categorical action by induction and restriction functors to the usual categorical action on modules over a KLRW algebra.

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1. Introduction

1.1. Categorification and affine Grassmannian slices

Let G_Γ be a semisimple complex group with Dynkin diagram Γ . In recent years, following groundbreaking ideas of Khovanov, there has been great interest in constructing categorifications of tensor product representations of G_Γ .

Khovanov-Lauda [KL09] and Rouquier [Rou] introduced a family of combinatorially defined diagrammatic algebras for this purpose. Their work was later extended by the second author, who defined KLRW algebras T_μ^λ , for a list $\lambda = (\lambda_1, \dots, \lambda_n)$ of dominant weights and a weight μ of G_Γ [Web17a]. The categories of modules over these algebras carry categorical \mathfrak{g}_Γ actions [Web17a, Theorem B.]. Recall that a categorical \mathfrak{g}_Γ action assigns a category to each weight space (in this case, T_μ^λ -mod) and a functor

$$\mathcal{E}_i : T_\mu^\lambda\text{-mod} \rightarrow T_{\mu+\alpha_i}^\lambda\text{-mod} \quad \mathcal{F}_i : T_\mu^\lambda\text{-mod} \rightarrow T_{\mu-\alpha_i}^\lambda\text{-mod} \quad (1.1)$$

for each $i \in I$. We require that $\bigoplus_\mu K_{\mathbb{C}}(T_\mu^\lambda\text{-mod}) \cong V(\lambda_1) \otimes \dots \otimes V(\lambda_n)$ as representations of \mathfrak{g}_Γ , with $\mathcal{E}_i, \mathcal{F}_i$ categorifying the Chevalley generators of \mathfrak{g}_Γ . Joint work of this author and Losev shows that these categories are the unique **tensor product categorification** for the tensor product above [LW15, Theorem A]. Thus, other categorifications of tensor products appearing in the literature will typically be equivalent to T_μ^λ -mod. The most notable of these is defined by translation functors on category \mathcal{O} , based on the original work of Bernstein-Frenkel-Khovanov [BFK99] in the case where $\mathfrak{g}_\Gamma = \mathfrak{sl}_2$.

In [KWWY14], we began a project to construct categorifications using affine Grassmannian slices and their quantizations. Affine Grassmannian slices $\overline{\mathcal{W}}_\mu^\lambda$ are defined for pairs of dominant weights λ, μ where $\mu \leq \lambda$. In [BFN19], these were generalized to arbitrary weights μ . These generalized affine Grassmannian slices are affine Poisson varieties with symplectic singularities [KWWY14, Wee22, Zho, Bel], defined using the affine Grassmannian of the Langlands dual group G_Γ^\vee . The main feature of these varieties is that they contain the Mirković-Vilonen cycles as the attracting loci of their fixed points [KWWY14, Kry18].

Generalized affine Grassmannian slices admit natural quantizations Y_μ^λ , called **truncated shifted Yangians**, defined for dominant μ in [KWWY14] and for general μ in [BFN19]. Motivated by the geometric Satake correspondence of Mirković-Vilonen [MV07] and the philosophy of Braden-Licata-Proudfoot-Webster [BLPW16], we conjectured in [KWWY14] that category \mathcal{O} for these algebras could be used to construct a categorification of tensor product representations. More precisely, these algebras appear as a family over a space of quantization parameters, which in this paper are called **flavours**; we let $Y_\mu^\lambda\text{-}\mathcal{O}_{\mathbb{Z}}$ be the category \mathcal{O} over this algebra for a generic integral flavour.

We expected the existence of exact functors

$$\mathcal{E}_i : Y_\mu^\lambda\text{-}\mathcal{O}_{\mathbb{Z}} \rightarrow Y_{\mu+\alpha_i}^\lambda\text{-}\mathcal{O}_{\mathbb{Z}} \quad \mathcal{F}_i : Y_\mu^\lambda\text{-}\mathcal{O}_{\mathbb{Z}} \rightarrow Y_{\mu-\alpha_i}^\lambda\text{-}\mathcal{O}_{\mathbb{Z}} \quad (1.2)$$

along with isomorphisms $\bigoplus_\mu K_{\mathbb{C}}(Y_\mu^\lambda\text{-}\mathcal{O}_{\mathbb{Z}}) \cong V(\lambda_1) \otimes \dots \otimes V(\lambda_n)$ of \mathfrak{g}_Γ -representations, for λ_k fundamental weights satisfying $\lambda = \lambda_1 + \dots + \lambda_n$.

In [KTW⁺19b], we made a decisive step in this direction by constructing equivalences of Abelian categories $Y_\mu^\lambda\text{-}\mathcal{O}_{\mathbb{Z}} \cong T_\mu^\lambda\text{-mod}$. In this way, we are able to define the functors (1.2) using the functors (1.1). However, this left the following question, which we will resolve in this paper:

Question 1.1. Can we define the functors (1.2) directly using truncated shifted Yangians?

1.2. Restriction functors for Coulomb branches

In this paper, we will construct these functors (1.2) by realizing them in the larger context of Coulomb branches. For any reductive group \tilde{G} , a representation N and a normal subgroup $G \subset \tilde{G}$, such that

$F = \tilde{G}/G$ (the flavour group) is a torus, Braverman-Finkelberg-Nakajima [BFN18] constructed an affine Poisson variety $M_C(G, N)$ and its deformation quantization $\mathcal{A}(G, N)$. In the notation of [BFN18], this second algebra would be denoted $\mathcal{A}_{\hbar}(G, N)/(\hbar - 1)$, but since we will be interested primarily in this noncommutative deformation or specializations of it, we will drop the subscript. We note that $\mathcal{A}(G, N)$ carries a filtration whose associated graded $\text{gr } \mathcal{A}(G, N)$ is isomorphic to the coordinate ring of $M_C(G, N)$. Following the physics literature:

- The variety $M_C(G, N)$ is called the (flavour deformation of the) **Coulomb branch** of the gauge theory defined by G, N .
- The algebra $\mathcal{A}(G, N)$ is called the **Coulomb branch algebra**.

An important feature of the Coulomb branch construction is that it comes with a complete integrable system $M_C(G, N) \rightarrow \tilde{\mathfrak{t}}/W \cong \tilde{\mathfrak{g}}//G$, where $\tilde{\mathfrak{t}}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}} = \text{Lie}(\tilde{G})$. The quantization of this integrable system yields a maximal commutative subalgebra $(\text{Sym } \tilde{\mathfrak{t}}^*)^W \subset \mathcal{A}(G, N)$, which we call the **Gelfand-Tsetlin subalgebra**, since it generalizes the usual Gelfand-Tsetlin subalgebra of $U(\mathfrak{gl}_n)$.

We consider modules over $\mathcal{A}(G, N)$, which are locally finite as $(\text{Sym } \tilde{\mathfrak{t}}^*)^W$ -modules. We call these **Gelfand-Tsetlin modules**, and let $\mathcal{A}(G, N)\text{-}\Gamma\text{II}$ be the category of Gelfand-Tsetlin modules (Definition 6.1). Modules M in this category have decompositions into generalized eigenspaces for the action of the Gelfand-Tsetlin algebra:

$$M = \bigoplus_{\gamma \in \tilde{\mathfrak{t}}/W} \mathcal{W}_{\gamma}(M).$$

Throughout this paper, we will only study modules on which the centre of $\mathcal{A}(G, N)$ acts semisimply, though we include discussion of how our results can be modified to account for more general modules.

The category of Gelfand-Tsetlin modules contains a category \mathcal{O} for $\mathcal{A}(G, N)$ -modules, denoted $\mathcal{A}(G, N)\text{-}\mathcal{O}$, consisting of modules satisfying a local-finiteness condition with respect to a \mathbb{C}^{\times} -action (cf. Definition 4.7). When G, N are constructed using ADE quiver data, then by [BFN19, Wee]:

- The variety $M_C(G, N)$ is a generalized affine Grassmannian slice.
- The algebra $\mathcal{A}(G, N)$ is a truncated shifted Yangian Y_{μ}^{λ} .

See Section 3.3 for the precise versions of these statements. Our main insight in this paper is that the relationship between Y_{μ}^{λ} and $Y_{\mu+\alpha_i}^{\lambda}$ can be studied as a special case of a more general ‘parabolic restriction’ of Coulomb branches.

Let $\xi : \mathbb{C}^{\times} \rightarrow T$ be a coweight of the group G . This determines a Levi subgroup L of G (its centralizer) and an L -subrepresentation N_0^{ξ} (the subspace of invariants for the cocharacter ξ). When G, N are chosen so that $\mathcal{A}(G, N) \cong Y_{\mu}^{\lambda}$ (Section 3.3), there is a particular choice of ξ (see Theorem 1.4) yielding $\mathcal{A}(L/\mathbb{C}_{\xi}^{\times}, N_0^{\xi}) = Y_{\mu+\alpha_i}^{\lambda}$. This inspired us to examine the relation between $\mathcal{A}(G, N)$ and $\mathcal{A}(L/\mathbb{C}_{\xi}^{\times}, N_0^{\xi})$ in the more general setting. Geometrically, $\dim M_C(L/\mathbb{C}_{\xi}^{\times}, N_0^{\xi}) = \dim M_C(G, N) - 2$ and so one might expect to obtain $M_C(L/\mathbb{C}_{\xi}^{\times}, N_0^{\xi})$ by Hamiltonian reduction of $M_C(G, N)$ by the action of an additive group. In the finite ADE quiver situation, it is possible to achieve this (see [KPW22] and Remark 3.2) but not in a way compatible with the above-mentioned integrable systems. Thus, in this paper, we pursue a different approach.

Our first main result (Theorem 2.14) describes the relationship between the four algebras $\mathcal{A}(G, N), \mathcal{A}(L, N), \mathcal{A}(L, N_0^{\xi})$ and $\mathcal{A}(L/\mathbb{C}_{\xi}^{\times}, N_0^{\xi})$. The exact statement is a bit complicated, but most importantly for our purposes, the maps between these algebras are compatible with their Gelfand-Tsetlin subalgebras, and allow us to prove the following result (Theorem 5.8).

Theorem 1.2. *There is a restriction functor $\mathcal{A}(G, N) - \Gamma\Pi \xrightarrow{\text{res}} \mathcal{A}(L, N_0^\xi) - \Gamma\Pi$, such that*

$$\mathcal{W}_v(\text{res}(M)) = \mathcal{W}_v(M)$$

for all $v \in \check{t}/W_L$ satisfying a condition called ξ -negative (which can be achieved by subtracting a sufficiently large integral multiple of ξ).

Finally, the algebra $\mathcal{A}(L/\mathbb{C}_\xi^\times, N_0^\xi)$ is the quantum Hamiltonian reduction of $\mathcal{A}(L, N_0^\xi)$ by the ‘monopole’ operator r_ξ . The Hamiltonian reduction functor also preserves the category of Gelfand-Tsetlin modules. The effect of this functor on weight spaces is described in Proposition 5.11.

1.3. Relating the functors

In this paper, we will consider any quiver Γ with vertex set I and edge set E . If Γ has no edge loops, then it has an associated symmetric Kac-Moody Lie algebra \mathfrak{g}_Γ . For the time being, we include the case where Γ has edge loops, but of course any statement that uses \mathfrak{g}_Γ can only apply in the case of no edge loops. Note that this is more general than our previous work [KTW⁺19b], where only simply-laced types with bipartite Dynkin diagrams are studied. We will write $M_C(\mathbf{v}, \mathbf{w})$ for the Coulomb branch, and $\mathcal{A}(\mathbf{v}, \mathbf{w})$ for the Coulomb branch algebra, defined using Γ and dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^I$. This means that we are using the gauge group and representation defined by

$$G = \prod_i GL(v_i) \quad N = \bigoplus_{i \rightarrow j \in E} \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_i \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i}).$$

We can consider the extended group $\tilde{G} = G \times (\mathbb{C}^\times)^E \times \prod_i (\mathbb{C}^\times)^{w_i}$, where the second and third factors act by scaling on the summands of N . When Γ is of ADE type, then $\mathcal{A}(\mathbf{v}, \mathbf{w})$ is isomorphic to Y_μ^λ , where we define $\lambda = \sum w_i \varpi_i$ and $\mu = \lambda - \sum v_i \alpha_i$ (see Section 3.3 for details).

We introduce a family of diagrammatic algebras which we call **flavoured KLRW algebras** $\tilde{\text{f}}\Gamma_{\mathbf{v}}^\varphi$, generalizing the metric KLRW algebras studied in [KTW⁺19b], and closely related to the weighted KLRW algebras introduced in [Web19b]. These depend on a choice of dimension vectors \mathbf{v}, \mathbf{w} , and also a choice of flavour φ : a choice of complex number for each edge of the Crawley-Boevey quiver $\Gamma^{\mathbf{w}}$ (cf. Section 3.1 for the definition of the Crawley-Boevey quiver). We write $\tilde{\text{f}}\Gamma_{\mathbf{v}}^\varphi\text{-wmod}$ for the category of weakly gradable modules.

By the Coulomb branch construction, we can also think of φ as a central character of $\mathcal{A}(\mathbf{v}, \mathbf{w})$. We generalize the main result from [KTW⁺19a] to give a precise description of the category $\mathcal{A}_\varphi(\mathbf{v}, \mathbf{w}) - \Gamma\Pi_{\mathbb{Z}}$ of Gelfand-Tsetlin modules over $\mathcal{A}(\mathbf{v}, \mathbf{w})$ with integral weights and integral central character φ (in the main body of the paper, we do not restrict to integral weights, but we do so in the introduction to make the statements simpler).

Theorem 1.3 (Corollary 9.6). *There is an equivalence of categories*

$$\mathcal{A}_\varphi(\mathbf{v}, \mathbf{w}) - \Gamma\Pi_{\mathbb{Z}} \cong \tilde{\text{f}}\Gamma_{\mathbf{v}}^\varphi\text{-wmod}.$$

Our answer to Question 1.1 is given by the following theorem.

Theorem 1.4 (Theorems 9.12 & 9.17). *Assume that Γ has no edge loops.*

1. *If we choose ξ to be the first fundamental coweight of $GL(v_i)$, the restriction functor from Theorem 1.2, combined with a Hamiltonian reduction, gives a functor*

$$\mathcal{A}_\varphi(\mathbf{v}, \mathbf{w}) - \Gamma\Pi_{\mathbb{Z}} \xrightarrow{\text{res}_i} \mathcal{A}_\varphi(\mathbf{v} - e_i, \mathbf{w}) - \Gamma\Pi_{\mathbb{Z}}.$$

2. We have a commutative diagram:

$$\begin{array}{ccc}
 \mathcal{A}_\varphi(\mathbf{v}, \mathbf{w}) - \Gamma\Pi_{\mathbb{Z}} & \xrightarrow{\text{res}_i} & \mathcal{A}_\varphi(\mathbf{v} - e_i, \mathbf{w}) - \Gamma\Pi_{\mathbb{Z}} \\
 \downarrow & & \downarrow \\
 \widetilde{\Gamma}_{\mathbf{v}}^\varphi\text{-wmod} & \xrightarrow{\mathcal{E}_i} & \widetilde{\Gamma}_{\mathbf{v}-e_i}^\varphi\text{-wmod}
 \end{array}, \tag{1.3}$$

where the vertical equivalences come from Theorem 1.3, and \mathcal{E}_i is a version of the functor (1.1)

3. The functors res_i preserve $\mathcal{A}_\varphi(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}}$, and with their adjoints ind_i give functors as in (1.2) which define a categorical \mathfrak{g}_{Γ} -action on $\bigoplus_{\mathbf{v}} \mathcal{A}_\varphi(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}}$.

1.4. Cherednik algebras

Similar induction and restriction functors were defined by Bezrukavnikov and Etingof [BE09, Section 3.5] in the context of category \mathcal{O} for rational Cherednik algebras. It's natural to compare these with the restriction and induction functors we define, since in the case where $G = GL(n)$ and $N = \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \text{Hom}(\mathbb{C}^n, \mathbb{C}^\ell)$, Kodera and Nakajima [KN18] (see also [BEF20, Web19a]) have shown that the Coulomb branch algebra $\mathcal{A}(n, \ell) = \mathcal{A}(G, N)$ is isomorphic to

- the spherical trigonometric Cherednik algebra $\mathfrak{H}(n, 0)$ of S_n if $\ell = 0$, and
- the spherical rational Cherednik algebra $\mathfrak{H}(n, \ell)$ of $G(\ell, 1, n) = S_n \wr \mathbb{Z}/\ell\mathbb{Z}$ otherwise.

In this paper, we will only consider the case $\ell > 0$. Let ξ denote the first fundamental coweight of $GL(n)$, as in Theorem 1.4. Then Theorem 2.14 yields a restriction functor $\text{res} : \mathcal{A}(n, \ell)\text{-}\mathcal{O} \rightarrow \mathcal{A}(n - 1, \ell)\text{-}\mathcal{O}$, while Bezrukavnikov and Etingof have defined a similar functor between category \mathcal{O} 's for the full Cherednik algebra $\mathfrak{H}(n, \ell)$. As shown in [GL14, Section 3.5.2], these preserve the subcategory of aspherical modules and thus induce functors $\text{res}_{\text{BE}} : \mathfrak{H}(n, \ell)\text{-}\mathcal{O} \rightarrow \mathfrak{H}(n - 1, \ell)\text{-}\mathcal{O}$, thought of as quotient categories of category \mathcal{O} for $\mathfrak{H}(n, \ell)$.

Note that unlike in Theorem 1.3, we are not assuming any integrality of parameters, since the most interesting cases for the Cherednik algebra involve rational but nonintegral parameters. On the contrary, we assume that $t \notin \mathbb{Z}$, though we expect that the result below is true (and, in fact, easier to prove) in the case where $t \in \mathbb{Z}$.

Theorem 1.5. *If $t \notin \mathbb{Z}$, there is an equivalence $\mathfrak{H}(n, \ell)\text{-}\mathcal{O} \rightarrow \mathcal{A}(n, \ell)\text{-}\mathcal{O}$ for each n which intertwines the functors res and res_{BE} .*

The proof of this will be given in Section 9.5. Given the discussion of the isomorphism $\mathfrak{H}(n, \ell) \cong \mathcal{A}(n, \ell)$ above, the reader might imagine that this is how the equivalence above is constructed. That is not, in fact, the case. Rather, the equivalence of Theorem 1.5 is constructed by comparing both categories to a flavoured KLRW algebra. This equivalence is far from unique; it depends on several choices, and, at the moment, it is not clear how to tweak all the choices involved to assure that we obtain the equivalence induced by the isomorphism of [KN18]. This is a question we will return to in future work.

Remark 1.6. Readers familiar with the rational Cherednik algebra might wonder about the relationship between Theorems 1.4 and 1.5, and the categorical action on category \mathcal{O} for rational Cherednik algebra constructed in [Sha11]. The reason we need to have $t \notin \mathbb{Z}$ is so that we can use Lemma 7.21 to show that when one compares $\mathcal{A}(n, \ell)\text{-}\mathcal{O}$ with the category $\mathcal{A}(\tilde{\mathbf{v}}, \tilde{\mathbf{w}})\text{-}\mathcal{O}$ for a different Coulomb branch $\mathcal{A}(\tilde{\mathbf{v}}, \tilde{\mathbf{w}})$ corresponding to a graph $\tilde{\Gamma}^{\mathbf{w}}$ with no loops; the equivalence of Theorem 1.5 will send Shan's action on $\mathfrak{H}(n, \ell)\text{-}\mathcal{O}$ to the action from Theorem 1.4 on $\mathcal{A}(\tilde{\mathbf{v}}, \tilde{\mathbf{w}})\text{-}\mathcal{O}$. The dependence on parameter is an artefact of a somewhat roundabout proof; however, there should be a proof of Theorem 1.5 which doesn't use Theorem 1.4 or Lemma 7.21.

1.5. Generalized geometric Satake

In [BFN19, Conjecture 3.25], Braverman, Finkelberg and Nakajima propose a generalization of the geometric Satake theorem to all symmetric Kac-Moody types, which is further developed in the affine type A case in [Nak]. For each character $G \rightarrow \mathbb{C}^\times$, there is an induced Hamiltonian \mathbb{C}^\times action on $M_C(\mathbf{v}, \mathbf{w})$. Choose this character to be given the product of the determinants, and let $\mathfrak{A}(\mathbf{v}, \mathbf{w}) \subset M_C(\mathbf{v}, \mathbf{w})$ be the attracting locus for this \mathbb{C}^\times action. If Γ is of finite ADE type, then $\mathfrak{A}(\mathbf{v}, \mathbf{w})$ is a Mirković-Vilonen locus in the affine Grassmannian of G_Γ^\vee by [Kry18, Lemma 4.4].

Conjecture 1.7 [BFN19, Conjecture 3.25(3)]. *The sum of the top Borel-Moore homologies $\bigoplus_{\mathbf{v}} H_{\text{top}}^{BM}(\mathfrak{A}(\mathbf{v}, \mathbf{w}))$ carries an action of \mathfrak{g}_Γ , making it isomorphic to the irreducible representation with the highest weight $\sum w_i \varpi_i$.*

In the finite-type case, this conjecture follows from the geometric Satake correspondence of Mirković-Vilonen [MV07]. It also holds in affine type A by the results of [Nak]. In general, we expect that $\dim \mathfrak{A}(\mathbf{v}, \mathbf{w}) = d$, where we define $d = \frac{1}{2} \dim M_C(\mathbf{v}, \mathbf{w})$; this is again known to hold in finite type by [MV07, Theorem 3.2], and affine type A by [NT17, Proposition 7.33].

This conjecture was an important motivation for us, since there is a close relationship between the category \mathcal{O} of $\mathcal{A}(\mathbf{v}, \mathbf{w})$ and the top homology appearing here. For any module M in category \mathcal{O} , there is an associated characteristic cycle class $\text{CC}(M) \in H_{2d}^{BM}(\mathfrak{A}(\mathbf{v}, \mathbf{w}))$. As in [BPW, Proposition 6.13], we can define this cycle by taking any good filtration of M (compatible with the above-mentioned filtration on $\mathcal{A}(G, N)$), and then summing the Borel-Moore classes of the d -dimensional components of $\text{gr } M$, weighted by the generic rank of $\text{gr } M$ on the component. This is well-defined by a standard argument of Bernstein [Ber, Lecture 2.8]; see also [Gin, Theorem 1.1.13]. Since we have fixed the degree of the Borel-Moore class here, this class will be insensitive to the multiplicity of our characteristic cycle on components of complex dimension $< d$.

This induces a map $K_{\mathbb{C}}(\mathcal{A}_\varphi(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}) \rightarrow H_{2d}^{BM}(\mathfrak{A}(\mathbf{v}, \mathbf{w}))$. This map is not an isomorphism in most cases, since if the Gelfand-Kirillov dimension of M is less than d , then $\text{CC}(M) = 0$ by Lemma 4.11. The kernel of this map also depends in a sensitive way on the choice of φ .

Assume that φ is integral, and let $\mathcal{O}_{\text{top}}(\mathbf{v}, \mathbf{w})$ be the quotient of $\mathcal{A}_\varphi(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}$ by the subcategory of objects with GK dimension $< d$. By Lemma 4.11, the characteristic cycle map descends to a map $K_{\mathbb{C}}(\mathcal{O}_{\text{top}}(\mathbf{v}, \mathbf{w})) \rightarrow H_{2d}^{BM}(\mathfrak{A}(\mathbf{v}, \mathbf{w}))$.

On the other hand, by Corollary 9.20, the sum $\bigoplus_{\mathbf{v}} K_{\mathbb{C}}(\mathcal{O}_{\text{top}}(\mathbf{v}, \mathbf{w}))$ is an irreducible representation of \mathfrak{g}_Γ , with the action induced by the induction and restriction functors $\mathcal{E}_i, \mathcal{F}_i$. Note, there is no dependence on φ in this result, beyond requiring it to be integral. This shows that, unlike the rest of the category \mathcal{O} , this quotient is not sensitive to φ . Thus, Conjecture 1.7 reduces to a proof that:

Conjecture 1.8. *We have an equality $\dim \mathfrak{A}(\mathbf{v}, \mathbf{w}) = d$ and for any integral φ , the characteristic cycle map $\bigoplus_{\mathbf{v}} K_{\mathbb{C}}(\mathcal{O}_{\text{top}}(\mathbf{v}, \mathbf{w})) \rightarrow \bigoplus_{\mathbf{v}} H_{2d}^{BM}(\mathfrak{A}(\mathbf{v}, \mathbf{w}))$ is an isomorphism of vector spaces.*

In finite-type ADE cases and in affine type A, the domain and codomain of this map both carry \mathfrak{g}_Γ -actions, but it is not clear that this map intertwines them. We know that the two sides are isomorphic as irreducible \mathfrak{g}_Γ -modules, so if the characteristic cycle map is equivariant, it must be an isomorphism. On the other hand, outside of the finite-type and affine type A cases, we have no preexisting action on the codomain, and thus we wish to use this conjecture to define one.

2. Relating various Coulomb branch algebras

2.1. Coulomb branch algebras and their partial flag versions

Let G be a reductive group. Let $T \subset B$ denote a fixed maximal torus and Borel subgroup, and let W denote the Weyl group. We write $\mathcal{K} = \mathbb{C}((z))$, $\mathcal{O} = \mathbb{C}[[z]]$, and we consider the groups $G(\mathcal{K}) = G((z))$ and $G(\mathcal{O}) = G[[z]]$. We will study the affine Grassmannian $\text{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$. Recall that for any dominant coweight λ , the $G(\mathcal{O})$ -orbits $\text{Gr}^\lambda = G(\mathcal{O})z^\lambda$ partition Gr_G .

Let P denote any parabolic subgroup of G containing T , and L its Levi subgroup. Let W_L be the Weyl group of L . There is a corresponding parahoric subgroup $I_P \subset G(\mathcal{O})$, which is defined as the preimage of P under the evaluation map $G(\mathcal{O}) \rightarrow G$. In particular, if $P = B$, then I_P is the usual Iwahori; on the other hand, if $P = G$, then $I_P = G(\mathcal{O})$. We have the corresponding partial affine flag variety $G(\mathcal{K})/I_P$.

We fix an extension

$$1 \rightarrow G \rightarrow \tilde{G} \rightarrow F \rightarrow 1,$$

where F is a torus. Following the physics literature, we call F the **flavour torus** of this extension.

Let \tilde{T} denote the maximal torus of \tilde{G} containing T . Similarly, for any parabolic subgroup P of G , let \tilde{P} be the parabolic subgroup in \tilde{G} , such that $G \cap \tilde{P} = P$ and \tilde{I}_P its preimage in $\tilde{G}(\mathcal{O})$. Note that the Levi \tilde{L} of \tilde{P} is an extension of F by L . We also have that $L(\mathcal{O}) \subset I_P$.

Throughout, we will regularly need to use the cohomology rings with complex coefficients of the classifying spaces of these groups, and so we write

$$\Lambda_G = H^*(BG) = H_G^*(pt) = (\text{Sym } \mathfrak{t}^*)^W,$$

and similarly for L, F, \tilde{G}, T , etc.

We define the group $\tilde{G}_{\mathcal{K}}^{\mathcal{O}}$ to be the preimage of $F(\mathcal{O})$ under the map $\tilde{G}(\mathcal{K}) \rightarrow F(\mathcal{K})$. This subgroup contains $\tilde{G}(\mathcal{O})$, and the quotient $\tilde{G}_{\mathcal{K}}^{\mathcal{O}}/\tilde{G}(\mathcal{O})$ is isomorphic to the affine Grassmannian Gr_G . We have an action of \mathbb{C}^\times on $\tilde{G}_{\mathcal{K}}^{\mathcal{O}}$ by loop rotation, given by $(t \cdot g)(z) = g(tz)$ for $t \in \mathbb{C}^\times$ and $g(z) \in \tilde{G}_{\mathcal{K}}^{\mathcal{O}}$. There is an action of the semidirect product $\tilde{G}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times$ on Gr_G .

Let N be a representation of \tilde{G} . We will typically construct \tilde{G} by starting with a representation of G , and letting \tilde{G} be the product of G and a torus which acts on N commuting with G ; for convenience later, we will not assume this action is faithful.

Following Braverman-Finkelberg-Nakajima [BFN18], we will now define the spaces used to construct the Coulomb branch. Let $N(\mathcal{O}) = N \otimes \mathbb{C}[[z]]$ and $N(\mathcal{K}) = N \otimes \mathbb{C}((z))$; these are naturally representations over $\tilde{G}(\mathcal{O}) \rtimes \mathbb{C}^\times$ and $\tilde{G}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times$, respectively. We can consider the vector bundle

$$T_{G,N} := G(\mathcal{K}) \times_{G(\mathcal{O})} N(\mathcal{O}) = \tilde{G}_{\mathcal{K}}^{\mathcal{O}} \times_{\tilde{G}(\mathcal{O})} N(\mathcal{O}) \rightarrow \text{Gr}_G.$$

Note that the subscript $G(\mathcal{O})$ (respectively, $\tilde{G}(\mathcal{O})$) indicates the quotient by a natural group action. There is a projection map $p : T_{G,N} \rightarrow N(\mathcal{K})$ given by $[g, v] \mapsto gv$.

Let

$$R_{G,N} = p^{-1}(N(\mathcal{O})) = \{([g], w) \in \text{Gr}_G \times N(\mathcal{O}) : w \in gN(\mathcal{O})\}.$$

Following Braverman-Finkelberg-Nakajima [BFN18], we form the convolution algebra

$$\mathcal{A}_{\hbar}(G, N) := H_*^{\tilde{G}(\mathcal{O}) \rtimes \mathbb{C}^\times}(R_{G,N}).$$

We will use \hbar throughout for the equivariant parameter corresponding to the loop \mathbb{C}^\times -action. We include it in the notation here to emphasize that we have left it as a free variable, whereas later, we will usually consider the quotient where we set $\hbar = 1$. A great deal of care is required to define the equivariant Borel-Moore homology for an infinite-dimensional space and group (see [BFN18, Section 2(ii)] for a detailed discussion). We will refer to $\mathcal{A}_{\hbar}(G, N)$ as the **spherical Coulomb branch algebra**.

Remark 2.1. If we specialize \hbar to 0, then the resulting algebra is commutative, and is the coordinate ring on a Poisson variety $M_C(G, N)$. The inclusion $H_F^*(pt) \rightarrow \mathcal{A}_{\hbar}(G, N)$ gives rise to a morphism $M_C(G, N) \rightarrow \mathfrak{f}$. The fibre over $0 \in \mathfrak{f}$ is usually called the **Coulomb branch**, and this family provides a Poisson deformation.

We will also need the partial flag variety version of the BFN algebra.

Definition 2.2. Let P be a parabolic subgroup of G . We define

$$\begin{aligned} T_{G,N}^P &:= G(\mathcal{K}) \times_{I_P} N(\mathcal{O}) = \tilde{G}_{\mathcal{K}}^{\mathcal{O}} \times_{\tilde{I}_P} N(\mathcal{O}) \xrightarrow{PP} N(\mathcal{K}) \\ R_{G,N}^P &:= p_P^{-1}(N(\mathcal{O})) = \{([g], w) \in (G(\mathcal{K})/I_P) \times N(\mathcal{O}) : w \in gN(\mathcal{O})\}. \end{aligned}$$

The **parabolic Coulomb branch algebra** is the convolution algebra

$$\mathcal{A}_{\hbar}^P(G, N) := H_*^{\tilde{I}_P \rtimes \mathbb{C}^\times}(R_{G,N}^P).$$

These algebras depend on the choice of \tilde{G} , but we leave this implicit in the notation. In the special case where $P = B$, then we will call \mathcal{A}_{\hbar}^B the **Iwahori Coulomb branch algebra**. The spherical Coulomb branch algebra is an example of a **principal Galois order**, as defined, that is in [FGRZ20]. The idea of replacing principal Galois orders by flag orders (defined in [Weba, Lemma 2.5]) played an important role in [Weba, Webb]; the algebra \mathcal{A}_{\hbar}^B is an example of a flag order in this case.

Note that the space $R_{G,N}^P$ contains a copy of $(G(\mathcal{O})/I_P) \times N(\mathcal{O}) \cong G/P \times N(\mathcal{O})$. We have vector space isomorphisms

$$H_*^{\tilde{I}_P \rtimes \mathbb{C}^\times}(G/P \times N(\mathcal{O})) \cong H_*^{\tilde{P} \times \mathbb{C}^\times}(G/P) \cong H_*^{\tilde{G} \times \mathbb{C}^\times}((G/P)^2) \tag{2.1}$$

(here, we use $G/P = \tilde{G}/\tilde{P}$ to get the action of \tilde{G} on G/P).

Now, $H_*^{\tilde{G} \times \mathbb{C}^\times}((G/P)^2)$ has a convolution structure of its own, and Poincaré duality shows that it is a matrix algebra:

$$H_*^{\tilde{G} \times \mathbb{C}^\times}((G/P)^2) \cong \text{End}_{\mathbb{A}_{\tilde{G} \times \mathbb{C}^\times}}(H_*^{\tilde{G} \times \mathbb{C}^\times}(G/P)) \cong \text{End}_{\mathbb{A}_{\tilde{G} \times \mathbb{C}^\times}}(\mathbb{A}_{\tilde{L} \times \mathbb{C}^\times}).$$

When $P = B$, then this is the nilHecke algebra of W .

Lemma 2.3. *The inclusion $(G(\mathcal{O})/I_P) \times N(\mathcal{O}) \hookrightarrow R_{G,N}^P$ induces an algebra map*

$$H_*^{\tilde{G} \times \mathbb{C}^\times}((G/P)^2) \rightarrow \mathcal{A}_{\hbar}^P(G, N).$$

Let $e' \in H_*^{\tilde{G} \times \mathbb{C}^\times}((G/P)^2) = H_*^{\tilde{P} \times \mathbb{C}^\times}(G/P)$ be the primitive idempotent in this matrix algebra that projects to the W -invariants. We can formulate the usual Abelianization isomorphism (see, for example, Proposition 1 in [Bri98]) as the statement that for any \tilde{G} -space X , we have $e'H_*^{\tilde{P}}(X) \cong H_*^{\tilde{G}}(X)$, where $H_*^{\tilde{P} \times \mathbb{C}^\times}(G/P)$ acts by convolution.

Applying this fact, we find that:

Proposition 2.4. *For any parabolic $P \subset G$, we have isomorphisms*

$$\mathcal{A}_{\hbar}^P e' \cong H_*^{\tilde{I}_P \rtimes \mathbb{C}^\times}(R_{G,N}) \quad e' \mathcal{A}_{\hbar}^P \cong H_*^{\tilde{G}(\mathcal{O}) \rtimes \mathbb{C}^\times}(R_{G,N}^P) \quad \mathcal{A}_{\hbar} \cong e' \mathcal{A}_{\hbar}^P e'. \tag{2.2}$$

The bimodules $\mathcal{A}_{\hbar}^P e'$ and $e' \mathcal{A}_{\hbar}^P$ define a Morita equivalence between \mathcal{A}_{\hbar}^P and \mathcal{A}_{\hbar} .

The first two isomorphisms of (2.2) are related by the map from \tilde{I}_P -orbits on $R_{G,N}$ to $\tilde{G}(\mathcal{O})$ orbits on $R_{G,N}^P$ sending $([g], v) \mapsto ([g^{-1}], g^{-1}v)$.

Proof. This follows the same proof as [Wee, Theorem 2.6]. □

The algebra $\mathcal{A}_{\hbar}(G, N)$ contains a subalgebra given by the equivariant parameters

$$H_{\tilde{G}(\mathcal{O}) \rtimes \mathbb{C}^\times}^*(pt) \cong \mathbb{A}_{\tilde{G} \times \mathbb{C}^\times}.$$

Braverman et al. [BFN18, Section 3(vi)] call this the Cartan subalgebra, but we prefer to call this the **Gelfand-Tsetlin subalgebra**, since it is a generalization of this subalgebra in $U(\mathfrak{gl}_n)$. Similarly, $\mathcal{A}_\hbar^P(G, N)$ contains $H_{I_P \rtimes \mathbb{C}^\times}^*(pt) \cong \wedge_{L \times \mathbb{C}^\times}$.

The Gelfand-Tsetlin algebra $\wedge_{\tilde{G} \times \mathbb{C}^\times}$ of $\mathcal{A}_\hbar(G, N)$ contains the subalgebra $Z := \wedge_{F \times \mathbb{C}^\times} \cong \text{Sym } \mathfrak{f}^*[\hbar]$, which is central in $\mathcal{A}_\hbar(G, N)$ (here, \mathfrak{f} is the Lie algebra of our flavour torus F). In fact, an application of [FO10, Theorem 4.1(4)], using the fact that $\mathcal{A}_\hbar(G, N)$ is a Galois order as shown in [Weba, Theorem B], shows that Z is the full centre of $\mathcal{A}_\hbar(G, N)$.

2.2. Inclusion of Coulomb branch algebras

One advantage of considering these parabolic Coulomb branch algebras is that they allow us to study the relation with Coulomb branch algebras defined by Levi subgroups.

The inclusion $L(\mathcal{K}) \subset G(\mathcal{K})$ gives rise to an inclusion $\text{Gr}_L \hookrightarrow G(\mathcal{K})/I_P$. Moreover, it is easy to see that the restriction of $R_{G,N}^P$ to Gr_L coincides with $R_{L,N}$. This leads to the following result.

Proposition 2.5. *There is an inclusion of algebras $\mathcal{A}_\hbar(L, N) \rightarrow \mathcal{A}_\hbar^P(G, N)$, which respects their Gelfand-Tsetlin subalgebras $\wedge_{L \times \mathbb{C}^\times}$.*

Proof. The inclusion $R_{L,N} \subset R_{G,N}^P$ gives an inclusion

$$\iota: H_*^{L(\mathcal{O}) \rtimes \mathbb{C}^\times}(R_{L,N}) \hookrightarrow H_*^{L(\mathcal{O}) \rtimes \mathbb{C}^\times}(R_{G,N}^P). \tag{2.3}$$

To see the compatibility of this inclusion with multiplication, we use an argument similar to that in [BFN18, Section 5(ii)]. Consider the analogue of the diagram [BFN18, (3.2)]:

$$R_{G,N}^P \times R_{L,N} \xleftarrow{p} p^{-1}(R_{G,N}^P \times R_{L,N}) \xrightarrow{q} q(p^{-1}(R_{G,N}^P \times R_{L,N})) \xrightarrow{m} R_{G,N}^P,$$

where $p: R_{G,N}^P \times L(\mathcal{K}) \rightarrow R_{G,N}^P \times T_{L,N}$ is given by $([g_2], w, g_1) \mapsto ([g_2], w, [g_1, w])$. This diagram defines a right action of $\mathcal{A}_\hbar(L, N)$ on $\mathcal{A}_\hbar^P(G, N)$. In fact, this diagram is a special case of the auxiliary action diagram (27) from [HKW], where $Z = R_{G,N}^P$ and $G = L$, except with all factors reversed. Thus, from [HKW, Proposition 4.13], we can deduce that it defines a right module structure of $\mathcal{A}_\hbar(L, N)$ on $\mathcal{A}_\hbar^P(G, N)$ commuting with left multiplication.

As in [BFN18, Lemma 5.7(1)], we can see that $1 \star b = \iota(b)$. More generally, we must have $a \star b = a\iota(b)$. This shows that for any $b, b' \in \mathcal{A}_\hbar(L, N)$, we have:

$$\iota(bb') = 1 \star (bb') = (1 \star b) \star b' = \iota(b) \star b' = \iota(b)\iota(b').$$

Thus, (2.3) is an algebra map.

The inclusion $L(\mathcal{O}) \subset I_P$ leads to an isomorphism

$$H_*^{L(\mathcal{O}) \rtimes \mathbb{C}^\times}(R_{G,N}^P) \cong H_*^{I_P \rtimes \mathbb{C}^\times}(R_{G,N}^P).$$

Composing this with (2.3) gives the desired injection. □

2.3. Abelian theories and monopole operators

Let $\nu: \mathbb{C}^\times \rightarrow T$ be a central coweight. Since ν is central, $z^\nu \in \text{Gr}_G$ is a $G(\mathcal{O})$ orbit. We let $r_\nu \in \mathcal{A}_\hbar(G, N)$ be the homology class of the preimage of z^ν in $R_{G,N}$. These elements r_ν are called **monopole operators**.

When G is Abelian, all coweights are central, and the elements r_ν form a basis for $\mathcal{A}_\hbar(G, N)$ as a left (or right) module over $\mathbb{A}_{\tilde{G} \times \mathbb{C}^\times}$, where ν runs over all coweights of G . Their relations are known explicitly by [BFN18, Section 4(iii)]:

$$r_\xi r_\nu = \prod_{\langle \mu, \xi \rangle > 0} \prod_{\langle \mu, \nu \rangle}^{d(\langle \mu, \xi \rangle, \langle \mu, \nu \rangle)} (\mu + (\langle \mu, \xi \rangle - j)\hbar) \prod_{\langle \mu, \xi \rangle < 0} \prod_{\langle \mu, \nu \rangle}^{d(\langle \mu, \xi \rangle, \langle \mu, \nu \rangle) - 1} (\mu + (\langle \mu, \xi \rangle + j)\hbar) r_{\xi + \nu}. \tag{2.4}$$

Here, the first and third products range over weights μ of N , with multiplicity. These are weights for the action of \tilde{G} , and the products above lie in the Gelfand-Tsetlin subalgebra $\mathbb{A}_{\tilde{G} \times \mathbb{C}^\times}$. Also $d : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ is the function defined by

$$d(a, b) = \begin{cases} 0, & \text{if } a, b \text{ have the same sign,} \\ \min\{|a|, |b|\}, & \text{if } a, b \text{ have different signs.} \end{cases}$$

It will also be useful to have an inverted version of these formulas:

$$r_\xi^{-1} r_\nu = \prod_{\langle \mu, \xi \rangle > 0} \prod_{\langle \mu, \nu - \xi \rangle}^{d(\langle \mu, \xi \rangle, \langle \mu, \nu - \xi \rangle)} \frac{1}{\mu - j\hbar} \prod_{\langle \mu, \xi \rangle < 0} \prod_{\langle \mu, \nu - \xi \rangle}^{d(\langle \mu, \xi \rangle, \langle \mu, \nu - \xi \rangle) - 1} \frac{1}{\mu + j\hbar} r_{\nu - \xi}. \tag{2.5}$$

This version follows by rearranging (2.4) for the product $r_\xi r_{\nu - \xi}$, using in addition the fact that for any weight $\mu \in \mathbb{A}_{\tilde{G} \times \mathbb{C}^\times}$, we have

$$r_\xi \mu = (\mu + \langle \mu, \xi \rangle \hbar) r_\xi$$

by [BFN18, (4.8)]. Note that this implies $r_\xi^{-1} \mu = (\mu - \langle \mu, \xi \rangle \hbar) r_\xi^{-1}$.

Remark 2.6. Note that equation (2.4) differs slightly from [BFN18], as we do not follow the convention from [BFN18, Section 2(i)] of shifting the weights of the loop rotation action on $N(\mathcal{K})$ by $1/2$.

We now recall some algebra homomorphisms defined in [BFN18], which will play an important role in this paper.

For general G , there is an Abelianization map $(\iota_*)^{-1} : \mathcal{A}_\hbar(G, N) \hookrightarrow \mathcal{A}_\hbar(T, N)_{loc}$ described in [BFN18, Remark 5.23]. Here, $\mathcal{A}_\hbar(T, N)_{loc}$ denotes the localization of $\mathcal{A}_\hbar(T, N)$ at the multiplicative set generated by $\hbar, \alpha + m\hbar$, where α runs over roots of G and $m \in \mathbb{Z}$.

Next, recall from [BFN18, Remark 5.14] that for any two representations N_1, N_2 , there is a natural injective map $\mathcal{A}_\hbar(G, N_1 \oplus N_2) \hookrightarrow \mathcal{A}_\hbar(G, N_2)$. For G Abelian, this map is given by [BFN18, Section 4(vi)]:

$$r_\nu \mapsto \prod_{\langle \mu, \nu \rangle < 0} \prod_{j = \langle \mu, \nu \rangle}^{-1} (\mu + j\hbar) r_\nu. \tag{2.6}$$

Here, the product is over the weights of N_1 . Also, since it is potentially confusing, we note that we have written r_ν for the monopole operators in $\mathcal{A}_\hbar(G, N_1 \oplus N_2)$ and $\mathcal{A}_\hbar(G, N_2)$, respectively.

Finally, assume there is a coweight $\wp : \mathbb{C}^* \hookrightarrow \tilde{T}$ which acts on N_2 by scalar multiplication of weight 1 and on N_1 with weight 0; this is always possible if we extend the flavour torus F . By [BFN18, Section 6(viii)], there is a ‘Fourier transform’ isomorphism

$$\mathcal{A}_\hbar(G, N_1 \oplus N_2) \xrightarrow{\sim} \mathcal{A}_\hbar(G, N_1 \oplus N_2^*), \tag{2.7}$$

which is the identity on the Gelfand-Tsetlin subalgebra $\mathbb{A}_{\tilde{G} \times \mathbb{C}^\times}$. In the Abelian case, this isomorphism is defined by the map

$$r_\nu \mapsto (-1)^{\delta(\nu)} r_\nu \quad \mu \mapsto \mu + \hbar \langle \mu, \varphi \rangle, \tag{2.8}$$

where $\delta(\nu) = \sum_{\langle \mu, \nu \rangle > 0} \langle \mu, \nu \rangle$. In this sum, μ ranges over weights of N_2 (if we use the conventions of [BFN18], as discussed in Remark 2.6, then this shift by φ is unnecessary).

For general G , the isomorphism is defined using the Abelian case, via the Abelianization map [BFN18, Lemmas 5.9–5.10].

2.4. Passing to invariants

For λ a dominant coweight of G , let $R_{G,N}^\lambda$ denote the preimage of the $G(\mathcal{O})$ -orbit closure $\overline{\text{Gr}^\lambda}$ under the map $R_{G,N} \rightarrow \text{Gr}_G$. We write $\mathcal{A}_\hbar(G, N)^\lambda$ for the subspace of the Coulomb branch algebra coming from the homology of $R_{G,N}^\lambda$. We can restrict the injective algebra map $\mathcal{A}_\hbar(G, N_1 \oplus N_2) \hookrightarrow \mathcal{A}_\hbar(G, N_2)$ to an injective linear map $\mathcal{A}_\hbar(G, N_1 \oplus N_2)^\lambda \hookrightarrow \mathcal{A}_\hbar(G, N_1)^\lambda$.

Lemma 2.7. *Let λ be a dominant coweight for G . Suppose that, for all $[g] \in \overline{\text{Gr}^\lambda}$, we have $g(N_2 \otimes \mathcal{O}) \subseteq N_2 \otimes \mathcal{O}$. Then we have $\mathcal{A}_\hbar(G, N_1 \oplus N_2)^\lambda = \mathcal{A}_\hbar(G, N_1)^\lambda$.*

Proof. The map $\mathcal{A}_\hbar(G, N_1 \oplus N_2)^\lambda \hookrightarrow \mathcal{A}_\hbar(G, N_1)^\lambda$ comes from pullback along the map

$$R_{G, N_1 \oplus N_2}^\lambda \rightarrow R_{G, N_1}^\lambda.$$

However, by the hypothesis, this map is a vector bundle (with fibre over $([g], w) \in R_{G, N_1}^\lambda$ being $g(N_2 \otimes \mathcal{O})$). □

Definition 2.8. For any coweight $\xi: \mathbb{C}^\times \rightarrow T$, we let $N_0^\xi, N_\pm^\xi, N_\geq^\xi, N_\leq^\xi$ be the sum of weight spaces where the weight of ξ is zero, positive, negative, nonnegative or nonpositive, respectively.

Note that we always have

$$N = N_-^\xi \oplus N_0^\xi \oplus N_+^\xi = N_\leq^\xi \oplus N_+^\xi = N_-^\xi \oplus N_\geq^\xi.$$

We will assume that the coweight ξ is central. Recall the monopole operator r_ξ from Section 2.3.

Theorem 2.9. *Assume that $N_-^\xi = 0$. The natural map $\mathcal{A}_\hbar(G, N) \hookrightarrow \mathcal{A}_\hbar(G, N_0^\xi)$ gives an isomorphism*

$$\mathcal{A}_\hbar(G, N)[r_\xi^{-1}] \cong \mathcal{A}_\hbar(G, N_0^\xi).$$

For the proof, we appeal to the following basic fact about Ore localizations:

Lemma 2.10. *Let S be a multiplicative set in a domain A . Suppose that $\varphi: A \hookrightarrow B$ is an injective homomorphism into a ring B , such that:*

- (i) *for all $s \in S$, the image $\varphi(s)$ is invertible in B ,*
- (ii) *for all $b \in B$, there exist $s \in S$ and $a \in A$ so that $b = \varphi(s)^{-1} \varphi(a)$.*

Then S satisfies the left Ore condition in A , and φ gives an isomorphism $S^{-1}A \cong B$.

Proof of Theorem 2.9. We will verify the stipulations of the lemma. Since \mathbb{C}^\times_ξ acts trivially on N_0^ξ , the element r_ξ is invertible in $\mathcal{A}_\hbar(G, N_0^\xi)$ (with inverse $r_{-\xi}$). Thus, it remains to check Condition (ii) above. This will be implied by the following statement:

(†) For all dominant λ , there exists $m \in \mathbb{Z}_{\geq 0}$, such that $r_\xi^m \mathcal{A}_\hbar(G, N_0^\xi)^\lambda \subset \mathcal{A}_\hbar(G, N)^{\lambda+m\xi}$.

Given a dominant coweight λ , let $m = \max\langle \mu, \lambda \rangle$, where μ ranges over all weights of the representation N_\pm^ξ . A standard reasoning shows that for all $[g] \in \overline{\text{Gr}^\lambda}$, we have $g(N_\pm^\xi \otimes \mathcal{O}) \subset z^{-m}(N_\pm^\xi \otimes \mathcal{O})$.

Thus, $z^{m\xi}g(N_+^\xi \otimes \mathcal{O}) \subset N_+^\xi \otimes \mathcal{O}$ (as the weights of \mathbb{C}_ξ^\times on N_+^ξ are positive). So by Lemma 2.7, $\mathcal{A}_\hbar(G, N)^{\lambda+m\xi} = \mathcal{A}_\hbar(G, N_0^\xi)^{\lambda+m\xi}$.

Now we observe that $r_\xi^m \mathcal{A}_\hbar(G, N_0^\xi)^\lambda = \mathcal{A}_\hbar(G, N_0^\xi)^{\lambda+m\xi}$ and thus we have established (†). □

2.5. Hamiltonian reduction

Note that the operations we’ve discussed thus far only relate groups with the same rank, so the dimension of the Coulomb branch will be unchanged. When we change the matter from N to N_0^ξ , the action of the central subgroup \mathbb{C}_ξ^\times becomes trivial. So we can consider N_0^ξ as a representation of the quotient group G/\mathbb{C}_ξ^\times . This invites us to consider the relationship between $\mathcal{A}_\hbar(G, N_0^\xi)$ and $\mathcal{A}_\hbar(G/\mathbb{C}_\xi^\times, N_0^\xi)$, which decreases the dimension of the Coulomb branch by 2.

Let A be an algebra, and $b \in A$. Recall that the **quantum Hamiltonian reduction** of A by b (at level 1) is defined in two stages. First, we form the right A -module $A/(b-1)A$. Then, we construct the quantum Hamiltonian reduction by considering

$$A \parallel_1 b := \text{End}_A(A/(b-1)A) \cong \{[a] \in A/(b-1)A \mid a(b-1) \in (b-1)A\}.$$

Thought of as an endomorphism ring, $A \parallel_1 b$ has a natural algebra structure, which may equivalently be defined on equivalence classes by $[a_1] \cdot [a_2] = [a_1 a_2]$. Geometrically, this algebra is the quantization of the operation of passing to the level set $b = 1$, and then dividing by the flow of the Hamiltonian vector field associated to b .

Theorem 2.11. *Assume that $\xi : \mathbb{C}^\times \rightarrow G$ is central and primitive (i.e. it is not an integer multiple of any other cocharacter), and that \mathbb{C}_ξ^\times acts trivially on N . Then we have*

$$\mathcal{A}_\hbar(G, N) \parallel_1 r_\xi \cong \mathcal{A}_\hbar(G/\mathbb{C}_\xi^\times, N).$$

Proof. For the purposes of this proof, let us write $G' = G/\mathbb{C}_\xi^\times$. Consider $R_{G',N}$. This space has an action of $G(\mathcal{O})$ which factors through the map $G(\mathcal{O}) \rightarrow G'(\mathcal{O})$. On the one hand,

$$H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(R_{G',N}) \cong \mathcal{A}_\hbar(G, N)/(r_\xi - 1)\mathcal{A}_\hbar(G, N)$$

as a module over $\mathcal{A}_\hbar(G, N)$. To see this, we note that $R_{G',N} = R_{G,N}/\mathbb{Z}$, where the generator of \mathbb{Z} acts by translation by the element z^ξ , since ξ is primitive.

On the other hand,

$$H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(R_{G',N}) \cong H_*^{G'(\mathcal{O}) \rtimes \mathbb{C}^\times}(R_{G',N}) \otimes_{\mathbb{A}_{G'}} \mathbb{A}_G$$

as a $\mathcal{A}_\hbar(G', N)$ module. Therefore, we have that

$$\mathcal{A}_\hbar(G, N)/(r_\xi - 1)\mathcal{A}_\hbar(G, N) \cong \mathcal{A}_\hbar(G', N) \otimes_{\mathbb{A}_{G'}} \mathbb{A}_G.$$

Note that $\mathbb{A}_G \cong \mathbb{A}_{G'}[a]$, where a is a character of the Lie algebra \mathfrak{g} splitting the derivative of ξ . The action of r_ξ on the RHS above commutes with $\mathbb{A}_{G'}$, and satisfies $[r_\xi, a] = r_\xi$. This shows that the kernel of $r_\xi - 1$ is $\mathcal{A}_\hbar(G', N)$. □

Remark 2.12. The hypothesis of primitivity is needed here. For example, if we reduce $\mathcal{A}_\hbar(\mathbb{C}^\times, 0)$ by the square $r_2 = r_1^2$, then the result will be $\mathbb{C}[r_1]/(r_1^2 - 1) \not\cong \mathbb{C}$.

Remark 2.13. One can also consider the ‘right’ quantum Hamiltonian reduction

$$\text{End}_A(A/A(b-1))^{op} \cong \{[a] \in A/A(b-1) \mid (b-1)a \in A(b-1)\}.$$

With the same assumptions as in the previous theorem, the right Hamiltonian reduction of $\mathcal{A}_\hbar(G, N)$ by r_ξ is also isomorphic to $\mathcal{A}_\hbar(G/\mathbb{C}_\xi^\times, N)$.

2.6. Combining all the steps

Now, we will see how to combine the above results.

Let G be a reductive group, N a representation, $\xi : \mathbb{C}^\times \rightarrow T$ any coweight. Let P, L be the parabolic and Levi subgroups corresponding to ξ , and let N_0^ξ be the invariants for the action of \mathbb{C}_ξ^\times on N .

Theorem 2.14. *The algebras*

$$\mathcal{A}_\hbar(G, N), \mathcal{A}_\hbar^P(G, N), \mathcal{A}_\hbar(L, N), \mathcal{A}_\hbar(L, N_0^\xi), \text{ and } \mathcal{A}_\hbar(L/\mathbb{C}_\xi^\times, N_0^\xi)$$

are related as follows.

1. There is a Morita equivalence between

$$\mathcal{A}_\hbar(G, N) \text{ and } \mathcal{A}_\hbar^P(G, N).$$

2. There is an inclusion of algebras

$$\mathcal{A}_\hbar(L, N) \hookrightarrow \mathcal{A}_\hbar^P(G, N).$$

3. There is an isomorphism

$$\mathcal{A}_\hbar(L, N)[r_\xi^{-1}] \cong \mathcal{A}_\hbar(L, N_0^\xi).$$

4. If ξ is primitive, there is an isomorphism

$$\mathcal{A}_\hbar(L, N_0^\xi) \parallel_1 r_\xi \cong \mathcal{A}_\hbar(L/\mathbb{C}_\xi^\times, N_0^\xi).$$

All these maps are compatible with the maps between the Gelfand-Tsetlin subalgebras

$$\mathbb{A}_{\tilde{G}} \hookrightarrow \mathbb{A}_{\tilde{L}} \hookrightarrow \mathbb{A}_{\tilde{L}/\mathbb{C}_\xi^\times}.$$

Proof.

1. This follows immediately from Proposition 2.4.
2. This follows immediately from Proposition 2.5.
3. Note that because ξ is central in L , all the subspaces from Definition 2.8 are invariant subspaces for the action of L . By (2.7), we have an isomorphism

$$\mathcal{A}_\hbar(L, N) \cong \mathcal{A}_\hbar(L, (N_-^\xi)^* \oplus N_\geq^\xi). \tag{2.9}$$

Since $N_-^{\xi*} \oplus N_\geq^\xi$ has nonnegative weight vectors for the action of \mathbb{C}_ξ^\times , we can apply Theorem 2.9 to give an isomorphism

$$\mathcal{A}_\hbar(L, N_-^{\xi*} \oplus N_\geq^\xi)[r_\xi^{-1}] \cong \mathcal{A}_\hbar(L, N_0^\xi).$$

Finally, we recall that map (2.9) is not the identity on the integrable system $\mathbb{A}_{\tilde{L}}$; it involves a shift by a cocharacter φ . However, this cocharacter acts trivially on N_0^ξ , and so $\mathcal{A}_\hbar(L, N_0^\xi)$ carries an automorphism shifting the integrable system $\mathbb{A}_{\tilde{L}}$ by $-\hbar\varphi$, and leaving all monopole operators

unchanged. Thus, if one considers the composition of the map (2.9) with the map of Theorem 2.9, and then finally the automorphism discussed above, the composition will give the desired map.

4. This follows immediately from Theorem 2.11. □

Remark 2.15. If we swap ξ and $-\xi$, then the algebras appearing in Theorem 2.14 are unchanged. However, the subspaces N_-^ξ, N_+^ξ are swapped, and so we will use a different map $\mathcal{A}_h(L, N) \rightarrow \mathcal{A}_h(L, N_0^\xi)$, which will induce a different isomorphism $\mathcal{A}_h(L, N)[r_{-\xi}^{-1}] \cong \mathcal{A}_h(L, N_0^\xi)$.

Remark 2.16. The assumption that ξ is primitive appears only in part (4) of the theorem. In fact, we may safely make this assumption throughout: for any ξ and any integer $k \geq 1$, the coweights ξ and $k\xi$ determine the same subgroups P, L , and determine the same subspaces $N_0^\xi = N_0^{k\xi}$ and $N_\pm^\xi = N_\pm^{k\xi}$. Moreover, $r_{k\xi} = r_\xi^k$ by (2.4), so $\mathcal{A}_h(L, N)[r_{k\xi}^{-1}] = \mathcal{A}_h(L, N)[r_\xi^{-1}]$. It follows that parts (1)–(3) of the theorem are identical for both ξ and $k\xi$, and for this reason we could assume that ξ is primitive throughout.

3. Quiver gauge theories

In this section, we focus on quiver gauge theories, and identify important special cases of the algebras which appear in Theorem 2.14.

3.1. Quiver data

Let Γ be a quiver with vertex set I and edge set $E(\Gamma)$, and dimension vectors $\mathbf{v}, \mathbf{w}: I \rightarrow \mathbb{Z}_{\geq 0}$ (we allow loops and multiple edges). For an edge $e \in E(\Gamma)$, we write $h(e)$ for its head and $t(e)$ for its tail.

Consider the group and representation

$$N = \bigoplus_{e \in E(\Gamma)} \text{Hom}(\mathbb{C}^{v_t(e)}, \mathbb{C}^{v_h(e)}) \oplus \bigoplus_{i \in I} \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i}) \quad G = \prod_{i \in I} GL(w_i).$$

We consider also the larger group

$$\tilde{G} := G \times \prod_{i \in I} (\mathbb{C}^\times)^{w_i} \times (\mathbb{C}^\times)^{E(\Gamma)},$$

where the middle factor is the product of the diagonal matrices inside each $GL(w_i)$. We have a (nonfaithful) action of \tilde{G} on N . We'll use $\mathcal{A}_h(\mathbf{v}, \mathbf{w}) := \mathcal{A}_h(G, N)$ to denote the Coulomb branch algebra attached to these dimension vectors.

This is slightly cleaner if we use the ‘Crawley-Boevey [CB01] trick’ of adding a new vertex ∞ and w_i new edges from i to ∞ to make a larger quiver Γ^w . We extend \mathbf{v} to this new vertex by setting $v_\infty = 1$. Then, we can think of an element of N as a representation of Γ^w , using $\text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i}) = \text{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_\infty})^{\oplus w_i}$, so $N = \bigoplus_{e \in E(\Gamma^w)} \text{Hom}(\mathbb{C}^{v_t(e)}, \mathbb{C}^{v_h(e)})$. Also, from this perspective, \tilde{G} is obtained from G by adding a scaling along each edge of Γ^w .

3.2. Relating algebras

As in the general case, we want to consider a coweight $\xi: \mathbb{C}^\times \rightarrow T$. This is the same as choosing a \mathbb{Z} -grading on the vector spaces \mathbb{C}^{v_i} ; to correct some later sign issues, we let the degree k elements be those with weight $-k$. Thus, for each $p \in \mathbb{Z}$, we have a dimension vector $v_i^{(p)}$, such that $v_i = \sum_{p \in \mathbb{Z}} v_i^{(p)}$.

In this case, L is the set of grading preserving automorphisms of these vector spaces, so $L = \prod_{p \in \mathbb{Z}} L^{(p)}$, where $L^{(p)} = \prod_{i \in I} GL(v_i^{(p)})$. The subspace N_0^ξ is just the grading preserving quiver representations, where \mathbb{C}^{w_i} is given degree 0. That is, $N_0^\xi = \bigoplus_{p \in \mathbb{Z}} N^{(p)}$, where

$$N^{(p)} = \begin{cases} \bigoplus_{e \in E(\Gamma)} \text{Hom}(\mathbb{C}^{v_r^{(p)}}, \mathbb{C}^{v_h^{(p)}}) & p \neq 0 \\ \bigoplus_{e \in E(\Gamma)} \text{Hom}(\mathbb{C}^{v_r^{(p)}}, \mathbb{C}^{v_h^{(p)}}) \oplus \bigoplus_i \text{Hom}(\mathbb{C}^{v_i^{(p)}}, \mathbb{C}^{w_i}) & p = 0. \end{cases}$$

By [BFN18, (3(vii).(a))], we immediately deduce the following.

Proposition 3.1. *We have an isomorphism of algebras*

$$\mathcal{A}_{\hbar}(L, N_0^{\xi}) \cong \cdots \otimes \mathcal{A}_{\hbar}(\mathbf{v}^{(-1)}, 0) \otimes \mathcal{A}_{\hbar}(\mathbf{v}^{(0)}, \mathbf{w}) \otimes \mathcal{A}_{\hbar}(\mathbf{v}^{(1)}, 0) \otimes \cdots,$$

where each factor on the RHS is again the Coulomb branch of a quiver gauge theory of the same underlying quiver, all but one of them unframed.

Thus, for a quiver gauge theory, Theorem 2.14 relates the algebra $\mathcal{A}_{\hbar}(\mathbf{v}, \mathbf{w})$ to the tensor product appearing in this proposition.

3.3. Truncated shifted Yangians

Assume that Γ is a Dynkin quiver, and let \mathfrak{g}_{Γ} be the corresponding simply-laced simple Lie algebra. The dimension vectors \mathbf{w}, \mathbf{v} , encode a pair of coweights λ, μ , for \mathfrak{g}_{Γ} , where $\lambda = \sum w_i \varpi_i$ and $\lambda - \mu = \sum v_i \alpha_i$.

Let Y_{μ}^{λ} denote the corresponding truncated shifted Yangian (with formal parameters and with \hbar) as defined in appendix B(viii) of [BFN19]. In [Wee], the fourth author proved that there is an isomorphism $\mathcal{A}_{\hbar}(\mathbf{v}, \mathbf{w}) \cong Y_{\mu}^{\lambda}$, building on the results in appendix B of [BFN19].

The splitting $v_i = \sum_{p \in \mathbb{Z}} v_i^{(p)}$ gives us a list of $\mu^{(p)}$, where $\mu^{(p)} = -\sum v_i^{(p)} \alpha_i$ for $p \neq 0$ and $\lambda - \mu^{(0)} = \sum v_i^{(0)} \alpha_i$ (in particular, we have $\mu = \sum \mu^{(p)}$). Thus, Theorem 2.14 and Proposition 3.1 relate Y_{μ}^{λ} with $Y_{\mu^{(0)}}^{\lambda} \otimes \bigotimes_{p \neq 0} Y_{\mu^{(p)}}^0$. For example, we can relate $Y_{\mu^{(0)}+\mu^{(1)}}^{\lambda}$ with $Y_{\mu^{(0)}}^{\lambda} \otimes Y_{\mu^{(1)}}^0$.

An important special case is when we take $\mu^{(1)} = -\alpha_i$ for some i and all other $\mu^{(p)} = 0$. This corresponds to taking ξ to be the i th standard coweight $\xi = \varpi_{i,1}$ (i.e. we map \mathbb{C}^{\times} into G by sending it to the upper left matrix slot of $GL(v_i)$). In this case, $(L/\mathbb{C}_{\xi}^{\times}, N_0^{\xi})$ is the quiver gauge theory corresponding to λ and $\mu + \alpha_i$, so $\mathcal{A}_{\hbar}(L/\mathbb{C}_{\xi}^{\times}, N_0^{\xi}) \cong Y_{\mu+\alpha_i}^{\lambda}$. Thus, Theorem 2.14 relates Y_{μ}^{λ} and $Y_{\mu+\alpha_i}^{\lambda}$. In Section 9.2, this will allow us to construct functors between category \mathcal{O} for these algebras, giving a categorical \mathfrak{g}_{Γ} -action.

Remark 3.2. In [FKP⁺ 18], half of the authors, along with Finkelberg, Pham and Rybnikov, constructed a comultiplication map $Y_{\mu^{(0)}+\mu^{(1)}} \rightarrow Y_{\mu^{(0)}} \otimes Y_{\mu^{(1)}}$. We believe that the comultiplication descends to maps $Y_{\mu^{(0)}+\mu^{(1)}}^{\lambda+\lambda^{(1)}} \rightarrow Y_{\mu^{(0)}}^{\lambda} \otimes Y_{\mu^{(1)}}^{\lambda^{(1)}}$, and, in particular, to a map $Y_{\mu^{(0)}+\mu^{(1)}}^{\lambda} \rightarrow Y_{\mu^{(0)}}^{\lambda} \otimes Y_{\mu^{(1)}}^0$. In [KPW22], we partially proved this statement in the case that $\mu^{(1)} = -\alpha_i$. However, we don't currently understand the relationship between comultiplication and Theorem 2.14. In particular, the comultiplication map is not compatible with the integrable systems.

4. Modules for Coulomb branch algebras

4.1. Gelfand-Tsetlin modules

We wish to understand the representation theory of the algebra $\mathcal{A}_{\hbar}(G, N)$, following the approach of [Webb, KTW⁺19b, Weba]. In particular, we will focus on the Gelfand-Tsetlin modules.

Now, and for the remainder of the paper, we specialize to $\hbar = 1$, and we will write $\mathcal{A}(G, N)$ for the result of this specialization¹. Later in the paper, we will further specialize at a point $\varphi \in \mathfrak{f} = \text{MaxSpec } Z$, and consider $\mathcal{A}_{\varphi}(G, N) = \mathcal{A}(G, N)/\mathcal{A}(G, N)_{\mathfrak{m}_{\varphi}}$, where $\mathfrak{m}_{\varphi} \subset Z$ is the corresponding maximal ideal.

¹In [KTW⁺19a, KTW⁺19b], we considered the specialization $\hbar = 2$ in order to match the conventions of some earlier papers. In any case, all nonzero \hbar give isomorphic specializations.

Given $\gamma \in \tilde{\mathfrak{t}}/W = \text{MaxSpec}(\mathbb{A}_{\tilde{G}})$, let $\mathfrak{m}_\gamma \subset \mathbb{A}_{\tilde{G}}$ be the corresponding maximal ideal. Consider the weight functors $\mathcal{W}_\gamma : \mathcal{A}(G, N)\text{-mod} \rightarrow \text{Vect}$ defined by

$$\mathcal{W}_\gamma(M) = \{m \in M \mid \mathfrak{m}_\gamma^N m = 0 \text{ for some } N \gg 0\}.$$

The reader might reasonably be concerned about the fact that this is a *generalized* eigenspace. In this paper, we will always want to consider these, and thus will omit ‘generalized’ before instances of ‘weight.’ These spaces $\mathcal{W}_\gamma(M)$ are called **Gelfand-Tsetlin (GT) weight spaces**.

Definition 4.1. An $\mathcal{A}(G, N)$ -module M is called a (*Z-semisimple*) **Gelfand-Tsetlin module** if it is finitely generated, $M = \bigoplus_{\gamma \in \tilde{\mathfrak{t}}/W} \mathcal{W}_\gamma(M)$, and the centre Z acts semisimply on M . For the rest of the paper, we let \mathcal{W}_γ denote the restriction of the weight functor to the category of Z -semisimple Gelfand-Tsetlin modules.

The **support** of a Gelfand-Tsetlin module is the set

$$\text{Supp}(M) = \{\gamma \in \tilde{\mathfrak{t}}/W \mid \mathcal{W}_\gamma(M) \neq 0\}.$$

Remark 4.2. In this paper, we will always assume that Gelfand-Tsetlin modules are Z -semisimple but will periodically add comments about how our results would change if we allowed the centre Z to act with nontrivial nilpotent part.

Remark 4.3. For the parabolic Coulomb branch algebra $\mathcal{A}^P(G, N)$, we also have a Gelfand-Tsetlin subalgebra $\mathbb{A}_{\tilde{L}}$. In a similar way, we can speak about Gelfand-Tsetlin weight functors ${}^P\mathcal{W}_\gamma$ and Gelfand-Tsetlin modules. In this case, the weight ν lies in $\tilde{\mathfrak{t}}/W_L = \text{MaxSpec}(\mathbb{A}_{\tilde{L}})$.

4.2. Morphisms between weight functors

Gelfand-Tsetlin modules can be classified using the approach of [DFO94], which is based on understanding the space of natural transformations $\text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'})$. Observe that

$$\text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'}) = \varprojlim \mathcal{A}/(\mathcal{A}\mathfrak{m}_\gamma^N + \mathfrak{m}_{\gamma'}^N \mathcal{A} + \mathcal{A}\mathfrak{m}_\varphi) = \varprojlim \mathcal{A}_\varphi/(\mathcal{A}_\varphi \mathfrak{m}_\gamma^N + \mathfrak{m}_{\gamma'}^N \mathcal{A}_\varphi),$$

for $\gamma, \gamma' \in \tilde{\mathfrak{t}}/W$ with common image $\varphi \in \mathfrak{f}$. If γ, γ' don’t have the same image in \mathfrak{f} , then $\text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'}) = 0$.

Remark 4.4. Note that this calculation of natural transformations is only valid for the restriction of \mathcal{W}_γ to the category of Z -semisimple modules; on the full category of $\mathcal{A}(G, N)$ -modules, each element $z \in Z$ defines a natural transformation. Under our convention of Z -semisimplicity, this element z acts on \mathcal{W}_γ by the scalar $\varphi(z)$, but there are Gelfand-Tsetlin modules which are not Z -semisimple where this does not hold.

The space $\text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'})$ has a natural weak operator topology, where a sequence converges if and only if it is eventually constant on $\mathcal{W}_\gamma(M)$ for all M . This is the same as the inverse limit topology.

These spaces $\text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'})$ can be organized into the algebra $\text{End}(\bigoplus \mathcal{W}_\gamma)$, or alternatively, into the category whose objects are weight functors. Since $\tilde{\mathfrak{t}}/W$ is an uncountably infinite set, $\text{End}(\bigoplus \mathcal{W}_\gamma)$ is a very large algebra. Luckily, it naturally decomposes into summands: consider the partition of $\tilde{\mathfrak{t}}/W$ into the disjoint union of the images of orbits of the extended affine Weyl group $\widehat{W} = N_{G(\mathcal{K})}(T)/T \cong W \ltimes \mathfrak{t}_Z$ on $\tilde{\mathfrak{t}}$. As discussed in [Weba, Section 2.5], the space of natural transformations $\text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'})$ is nonzero if and only if γ and γ' lie in the image $\bar{\mathcal{S}} \subset \tilde{\mathfrak{t}}/W$ of a \widehat{W} -orbit $\mathcal{S} \subset \tilde{\mathfrak{t}}$. Thus, an indecomposable Gelfand-Tsetlin module must have weights concentrated on the image of single orbit \mathcal{S} and the category of all Gelfand-Tsetlin modules decomposes as a direct sum subcategories of modules with weights concentrated on the image of single orbit.

Consider a set $S \subset \tilde{\mathfrak{t}}$, and its image $\bar{S} \subset \tilde{\mathfrak{t}}/W$. We define² $\mathcal{A}(G, N) - \Gamma\Pi_S$ to be the category of Gelfand-Tsetlin modules modulo the subcategory of modules killed by \mathcal{W}_γ for all $\gamma \in \bar{S}$. In the case where $S = \mathcal{S}$ is a single \widehat{W} -orbit, $\mathcal{A}(G, N) - \Gamma\Pi_S$ is just the subcategory of modules with support in \mathcal{S} .

Consider the algebra $F(S) = \text{End}(\bigoplus_{\gamma \in \bar{S}} \mathcal{W}_\gamma)$. More generally, given two sets $S, S' \subset \tilde{\mathfrak{t}}$, we let

$$F(S, S') = \text{Hom} \left(\bigoplus_{\gamma \in \bar{S}} \mathcal{W}_\gamma, \bigoplus_{\gamma' \in \bar{S}'} \mathcal{W}_{\gamma'} \right), \tag{4.1}$$

which is an $F(S'), F(S)$ bimodule.

Proposition 4.5 [DFO94, Theorem 17]; [Weba, Theorem 2.23]. *The functor $\mathcal{W}_S := \bigoplus_{\gamma \in \bar{S}} \mathcal{W}_\gamma$ gives an equivalence of categories between $\mathcal{A}(G, N) - \Gamma\Pi_S$ and modules over $F(S)$ continuous in the discrete topology.*

We recall the following definition from [Weba, Definition 2.24].

Definition 4.6. Let $\mathcal{S} \subset \tilde{\mathfrak{t}}$ be a \widehat{W} -orbit. A finite set $S \subset \mathcal{S}$ is called a **complete set** for \mathcal{S} , if for every $\gamma \in \mathcal{S}$, there is a $\gamma' \in S$, such that $\mathcal{W}_\gamma \cong \mathcal{W}_{\gamma'}$.

By [Weba, Corollary 4.16], a complete set S always exists. In this case, $\mathcal{A}(G, N) - \Gamma\Pi_S$ is equal to $\mathcal{A}(G, N) - \Gamma\Pi_{\mathcal{S}}$. Moreover, if S, S' are both complete sets for \mathcal{S} , then $F(S, S')$ gives a Morita equivalence between $F(S)$ and $F(S')$. Note that the existence of a complete set shows that $\mathcal{A}(G, N) - \Gamma\Pi_S$ is Artinian, since the functor \mathcal{W}_S is an equivalence and any module in its image is finite-dimensional.

4.3. Category \mathcal{O}

The Coulomb branch $\mathcal{A}(G, N)$ has a Hamiltonian action of the torus $K = (G/[G, G])^\vee$, so we have a Hamiltonian \mathbb{C}^\times -action on $\mathcal{A}(G, N)$ for each character χ of G .

More precisely, the map $\chi : G \rightarrow \mathbb{C}^\times$ induces a map $\text{Gr}_G \rightarrow \text{Gr}_{\mathbb{C}^\times}$. Since $\text{Gr}_{\mathbb{C}^\times} = \mathbb{Z}$, this gives a \mathbb{Z} -grading to $\mathcal{A}(G, N)$, and thus an action of \mathbb{C}^\times . By abuse of notation, we let χ also denote the derivative of the character χ , interpreted as an element of $(\mathfrak{t}^*)^W$. Then by [BFN18, Lemma 3.19], we have the quantum moment map relation: for $a \in \mathcal{A}(G, N)$ an element of weight $k \in \mathbb{Z}$ (with respect to χ), we have $[\chi, a] = ka$.

Definition 4.7. The **category** $\mathcal{A}(G, N) - \mathcal{O}$ for χ is the full subcategory of finitely generated $\mathcal{A}(G, N)$ -modules on which

- \mathbb{Z} acts semisimply, and
- χ acts locally finitely, with finite-dimensional generalized eigenspaces with eigenvalues whose real parts are bounded above.

Remark 4.8. A module in $\mathcal{A}(G, N) - \mathcal{O}$ is necessarily a Gelfand-Tsetlin module, so this is equivalent to asking that elements of nonnegative weight for the grading induced by χ act locally finitely (see [BLPW16, Lemma 3.13]).

As discussed in [Webb, Theorem 4.10], we can realize category $\mathcal{A}(G, N) - \mathcal{O}$ using the same algebraic approach that we used to understand the category $\mathcal{A}(G, N) - \Gamma\Pi_S$. The set $\bar{\mathcal{S}}$ is divided into finitely many equivalence classes by the relation $\gamma \sim \gamma'$ if $\mathcal{W}_\gamma \cong \mathcal{W}_{\gamma'}$ as functors. On each equivalence class, either:

- Thought of as a function on $\bar{\mathcal{S}}$, the function $\gamma \mapsto \mathfrak{R}(\chi(\gamma))$ attains a maximum, and attains it at a finite number of points. In this case, we call the equivalence class **bounded**.
- On $\bar{\mathcal{S}}$, the function $\gamma \mapsto \mathfrak{R}(\chi(\gamma))$ either has no maximum, or it attains its maximum on an infinite set. In this case, we call the equivalence class **unbounded**.

²Since English-speaking readers may not be familiar with these letters, the cyrillic $\Gamma\Pi$ is pronounced roughly ‘geh-tseh.’ These are the first letters of the names ‘Gelfand’ and ‘Tsetlin’ in Russian.

Proposition 4.9. *Every module in category $\mathcal{A}(G, N)\text{-}\mathcal{O}$ is a Gelfand-Tsetlin module. Conversely, a Gelfand-Tsetlin module $M \in \mathcal{A}(G, N)\text{-}\Gamma\Pi_{\mathcal{S}}$ lies in category $\mathcal{A}(G, N)\text{-}\mathcal{O}$ if and only if $\mathcal{W}_{\gamma}(M) = 0$ for all γ in an unbounded equivalence class.*

Thus, let $\mathcal{A}(G, N)\text{-}\mathcal{O}_{\mathcal{S}}$ denote the subcategory of $\mathcal{A}(G, N)\text{-}\Gamma\Pi_{\mathcal{S}}$ given by objects in $\mathcal{A}(G, N)\text{-}\mathcal{O}$. For any $S \subset \tilde{\mathfrak{t}}$, let $\mathcal{I} \subset F(S)$ be the two-sided ideal generated by the identity on \mathcal{W}_{γ} for all $\gamma \in \tilde{S}$ whose equivalence class is unbounded (alternatively, this is the span of the natural transformations that factor through such \mathcal{W}_{γ}).

This gives us the following modification of Proposition 4.5:

Proposition 4.10. *Let $S \subset \mathcal{S}$ be a finite complete set. The functor $\oplus_{\gamma \in \tilde{S}} \mathcal{W}_{\gamma}$ gives an equivalence of categories between $\mathcal{A}(G, N)\text{-}\mathcal{O}_{\mathcal{S}}$ and modules over $F(S)/\mathcal{I}$ continuous in the discrete topology.*

As mentioned above, this is a rephrasing of [Webb, Theorem 4.10]: in the notation of that paper, the algebra $F(S)$ would be denoted A_S and the ideal \mathcal{I} by $\mathcal{I}_{-\chi}$.

4.4. Gelfand-Kirillov dimension

Let \mathbb{k} be a field. Consider a \mathbb{k} -algebra A which is generated by a finite-dimensional subspace A_0 , and a left A -module M which is finitely generated by a finite-dimensional subspace M_0 . In this context, the **Gelfand-Kirillov dimension** $\text{GKdim}_A(M)$ is defined by:

$$\text{GKdim}_A(M) = \limsup_{n \rightarrow \infty} \log_n \dim_{\mathbb{k}}(A_0^n M_0). \tag{4.2}$$

It's a standard result that this number is independent of the choice of A_0 and M_0 , and only depends on the structure of M as an A -module.

Given a Gelfand-Tsetlin module M for $\mathcal{A}(G, N)$, let M^B denote the corresponding module of the Morita equivalent algebra $\mathcal{A}^B(G, N)$ (see Theorem 2.14(1) and Section 5.1 below). Recall that $\mathcal{A}^B(G, N)$ contains $\mathcal{A}(T, N)$ as a subalgebra, by Proposition 2.5.

Lemma 4.11. *Let M be a Gelfand-Tsetlin module for $\mathcal{A}(G, N)$. Consider:*

1. *The Gelfand-Kirillov dimension of M over $\mathcal{A}(G, N)$.*
2. *The Gelfand-Kirillov dimension of M^B over $\mathcal{A}(T, N)$.*
3. *The dimension of the Zariski closure of the support of M in $\tilde{\mathfrak{t}}/W$.*
4. *The Krull dimension of $\text{gr } M$ as a $\text{gr } \mathcal{A}(G, N)$ module, for any good filtration of M .*

Then (1) = (2) = (3) \geq (4).

Proof. (1) \geq (2): This is a consequence of:

$$\text{GKdim}_{\mathcal{A}(G, N)}(M) = \text{GKdim}_{\mathcal{A}^B(G, N)}(M^B) \geq \text{GKdim}_{\mathcal{A}(T, N)}(M^B).$$

Here, the equality on the left is due to the Morita equivalence of $\mathcal{A}(G, N)$ and $\mathcal{A}^B(G, N)$, while the inequality on the right comes simply from the fact that $\mathcal{A}(T, N) \subset \mathcal{A}^B(G, N)$ is a subalgebra.

(2) = (3): This follows from the same argument as [MV98, Proposition 8.2.3]. In fact, if the action of T on N is faithful, then by [BFN18, Section 4(vii)], the algebra $\mathcal{A}(T, N)$ is precisely one of the algebras B^{χ} considered by [MV98].

(3) \geq (1): As noted above, we may equivalently look at the Gelfand-Kirillov dimension of M^B over $\mathcal{A}^B(G, N)$. Now, if $a \in \mathcal{A}^B(G, N)$ is any element, then there exist elements $w_1, \dots, w_p \in \widehat{W}$ of the extended affine Weyl group, such that

$$a \cdot \mathcal{W}_{\lambda}(M^B) \subseteq \sum_{i=1}^p \mathcal{W}_{w_i(\lambda)}(M^B) \tag{4.3}$$

for every weight $\lambda \in \tilde{\mathfrak{t}}$. This claim follows from the results of [Weba] (see also [FO14, Lemma 3.1(4)]). Explicitly, the localization of $\mathcal{A}^B(G, N)$ at the fraction field of $\mathbb{A}_{\tilde{\mathfrak{T}}}$ is isomorphic to the smash product $\text{Frac}(\mathbb{A}_{\tilde{\mathfrak{T}}}) \# \widehat{W}$. The image $a \mapsto \sum_{i=1}^p a_i w_i$ in this localization is then a finite sum, where $a_i \in \text{Frac}(\mathbb{A}_{\tilde{\mathfrak{T}}})$ and $w_i \in \widehat{W}$. These are precisely the desired elements w_i .

Choose any finite-dimensional generating subspace A_0 of $\mathcal{A}^B(G, N)$, which exists by [BFN18, Proposition 6.8]. Then we can choose $w_1, \dots, w_p \in \widehat{W}$, such that (4.3) holds uniformly for all $a \in A_0$.

Now, fix any W -invariant norm on $\tilde{\mathfrak{t}}$. Note that each w_i above is given by an element of the finite Weyl group times a translation v_i . Let ϵ be the maximum of the norms of these elements v_i . Thus, by the triangle inequality, the ball $B(t)$ of radius t around 0 satisfies $w_i \cdot B(t) \subset B(t + \epsilon)$ for any $t > 0$. Consequently, by (4.3), we have

$$A_0 \cdot \bigoplus_{\lambda \in B(t)} \mathcal{W}_\lambda(M^B) \subset \bigoplus_{\lambda \in B(t+\epsilon)} \mathcal{W}_\lambda(M^B).$$

Next, choose any finite-dimensional generating set M_0 for M^B . It is contained in a sum of finitely many weight spaces, so we can choose t_0 so that $M_0 \subset \bigoplus_{\lambda \in B(t_0)} \mathcal{W}_\lambda(M^B)$. Thus, we find that

$$\dim A_0^n M_0 \leq \sum_{\lambda \in B(t_0+n\epsilon)} \dim \mathcal{W}_\lambda(M^B).$$

We conclude that the Gelfand-Kirillov dimension satisfies:

$$\text{GKdim}_{\mathcal{A}^B(G, N)}(M^B) \leq \lim_{t \rightarrow \infty} \frac{\log \sum_{\lambda \in B(t)} \dim \mathcal{W}_\lambda(M^B)}{\log t}.$$

Consider the equivalence relation $\lambda \sim \lambda'$ iff $\mathcal{W}_\lambda \cong \mathcal{W}_{\lambda'}$. By [Weba, Corollary 4.16], each \widehat{W} -orbit only contains finitely many equivalence classes, called clans, and each is the W -orbit of an intersection of a \mathfrak{t}_Z -coset with a convex polyhedron. Each indecomposable Gelfand-Tsetlin module is supported on a single \widehat{W} orbit (this follows from (4.3); see also the discussion before Proposition 4.5), so by finite generation, $\text{Supp}(M^B)$ is contained in a finite union of \widehat{W} -orbits. Thus, there are only finitely many equivalence classes $\{U_1, \dots, U_r\}$ in $\text{Supp}(M^B)$.

The Zariski closure $\overline{\text{Supp}(M^B)}$ is thus the union of the Zariski closures $\overline{U_i}$. Thus, if we define $d = \dim \overline{\text{Supp}(M^B)}$, then $d = \max_{i=1}^r \dim(\overline{U_i})$. Similarly, we have an equality

$$\lim_{t \rightarrow \infty} \frac{\log \dim \bigoplus_{\lambda \in B(t)} \mathcal{W}_\lambda(M^B)}{\log t} = \max_{i=1}^r \lim_{t \rightarrow \infty} \frac{\log |U_i \cap B(t)| + \log m_i}{\log t},$$

where $m_i = \dim \mathcal{W}_\lambda(M^B)$ for any $\lambda \in U_i$ (these dimensions are constant by the definition of U_i).

Since U_i is a finite union of the intersections of \mathfrak{t}_Z -cosets with convex polyhedra, this latter growth rate is exactly the dimension of its Zariski closure. This shows that

$$\lim_{t \rightarrow \infty} \frac{\log \dim \bigoplus_{\lambda \in B(t)} \mathcal{W}_\lambda(M^B)}{\log t} = d$$

which completes the proof that (3) \geq (1).

Finally, (1) \geq (4) is a general property of passing to associated graded algebras and modules. Indeed, if the Krull dimension of $\text{gr } M$ is q , then by Noether normalization, there are q algebraically independent elements $\bar{a}_1, \dots, \bar{a}_q$ of $\text{gr } \mathcal{A}(G, N)$, and which act on some element $\bar{v} \in \text{gr } M$ with no relations beyond commutativity. Thus, if v, a_1, \dots, a_q are preimages of these elements, and A_0 the span of a_1, \dots, a_q , then we have $\dim A_0^n \cdot v \geq \binom{n+q-1}{q-1}$, so this shows that $\text{GKdim}_{\mathcal{A}(G, N)}(M) \geq q$. \square

Let us include here a closely related result:

Lemma 4.12. *Let M be a Gelfand-Tsetlin module for $\mathcal{A}(G, N)$. Then*

$$\text{GK-dim}(\mathcal{A}(G, N)/\text{ann}(M)) = 2 \text{GK-dim}(M).$$

Proof. In the case $G = T$, this proven by Musson and van der Bergh in [MV98, Corollary 8.2.5].

Thus, in the general case, exactly as in the proof of Lemma 4.11, we can consider M^B over the Morita equivalent algebra $\mathcal{A}^B(G, N)$. Let $A = \mathcal{A}^B(G, N)/\text{ann}_{\mathcal{A}^B(G, N)}(M^B)$ and $A_T = \mathcal{A}(T, N)/\text{ann}_{\mathcal{A}(T, N)}(M^B)$. The Abelian case proves that

$$\text{GK-dim}(A_T) = 2 \text{GK-dim}(M^B),$$

where the latter is regarded as an $\mathcal{A}(T, N)$ module. By the previous Lemma, we have $\text{GK-dim}(M) = \text{GK-dim}(M^B)$ and so it suffices to prove that

$$\text{GK-dim}(A) = \text{GK-dim}(A_T).$$

Since $\mathcal{A}(T, N)/\text{ann}_{\mathcal{A}(T, N)}(M^B)$ is a subalgebra of $\mathcal{A}^B(G, N)/\text{ann}(M^B)$, we thus have

$$\text{GK-dim}(A) \geq \text{GK-dim}(A_T).$$

Thus, we need only prove the opposite inequality. For $w \in \widehat{W}$, let $R_{G, N}^B(\leq w)$ be the preimage of the Schubert variety \overline{IwI}/I in $R_{G, N}^B$. Let $\mathcal{A}^B(\leq w)$ be the homology of this subspace; this is a $\mathbb{A}_T - \mathbb{A}_T$ -subbimodule of $\mathcal{A}^B(G, N)$. Consider $\mathcal{A}^B(\leq w)/\mathcal{A}^B(< w)$. This quotient is the homology $H_*^{I \times \mathbb{C}^\times}(IwI/I)$, which is a free module of rank 1 over \mathbb{A}_T as a left module or as a right module, with the two actions differing by the action of w . The image of \mathbb{A}_T in A is a quotient of this polynomial ring. Note that for a simple $\mathcal{A}(T, N)$ GT-module M' , the kernel of the map $\mathbb{A}_T \rightarrow \text{ann}_{\mathcal{A}(T, N)}(M')$ is the intersection of the maximal ideals associated to the nonzero weight spaces, and is thus the radical ideal of polynomials vanishing on $\text{Supp}(M')$; for example, this follows from [MV98, Proposition 3.1.7].

This implies that the kernel for an arbitrary $\mathcal{A}(T, N)$ GT-module lies in the product of finitely many such ideals. In particular, the image of the map $\mathbb{A}_T \rightarrow A$ is a not-necessarily-radical quotient, whose support as a coherent sheaf is $\text{Supp}(M^B)$.

Thus, the same is true of any subquotient of A as a left or right \mathbb{A}_T -module, considered as a quasi-coherent sheaf. In particular, taking the corresponding quotient $A(\leq w)/A(< w)$, we thus obtain a $\mathbb{A}_T - \mathbb{A}_T$ -bimodule whose support as a left and a right module must be in $\overline{\text{Supp}(M^B)}$. Since these actions differ by w , the support as a left \mathbb{A}_T module must lie in $\overline{\text{Supp}(M^B)} \cap w \cdot \overline{\text{Supp}(M^B)}$. Now, note that all the components of $\overline{\text{Supp}(M^B)}$ are affine subspaces which are $\leq d$ -dimensional, since the Zariski closure of a clan has this form; now, for any two affine subspaces E_1, E_2 of an n -dimensional affine space, modelled on vector subspaces V_1, V_2 which are m_1 - and m_2 -dimensional. The set of elements x , such that $E_1 \cap E_2 \neq \emptyset$ is an affine subspace itself, modelled on the span $V_1 + V_2$, and this intersection is an affine subspace modelled on $V_1 \cap V_2$. If we apply this with E_1 , a component of $\overline{\text{Supp}(M^B)}$, and E_2 , the image of such a component under an element of the finite Weyl group, we find that the intersection $\overline{\text{Supp}(M^B)} \cap w \cdot \overline{\text{Supp}(M^B)}$ is $\geq k$ -dimensional only for extended affine Weyl group elements whose translation parts lie in a $2d - k$ -dimensional variety, where $d = \dim \text{Supp}(M^B)$.

Choose a generating set of the extended affine Weyl group containing the simple reflections, let $\ell_0(w) \geq \ell(w)$ be the length function of with respect to these generators (since G is not usually semisimple, we will need extra generators). Let $\mathcal{A}^B(\leq n)$ be the span of $\mathcal{A}^B(\leq w)$ for all elements w with $\ell_0(w) \leq n$. Now, consider the span \mathcal{A}_0 of

1. the degree 1 elements $\mathfrak{t}^* \subset \mathbb{A}_T$ and
2. generators of $\mathcal{A}^B(\leq n)$ as a left \mathbb{A}_T -module for some large n .

For n sufficiently large, this will be a set of generators of $\mathcal{A}^B(G, N)$ as an algebra. Let A_0 be the image of \mathcal{A}_0 in A . We need to show that the dimension of A_0^m does not grow too quickly in terms of m ; in order to obtain this bound, we use the filtration $A(\leq w)$ discussed above, and the loop filtration $F_p\mathcal{A}$ induced by the homological grading before we specialize \hbar to 1.

In particular, we let D be the smallest integer, such that $\mathcal{A}_0 \subset F_D\mathcal{A}^B$.

Thus, we have $\mathcal{A}_0^m \subset \mathcal{A}^B(\leq nm) \cap F_{Dm}\mathcal{A}$.

If $\dim \overline{\text{Supp}(M^B)} \cap w \cdot \overline{\text{Supp}(M^B)} \leq k$, then we must have that the dimension of $F_pA(\leq w)/F_pA(< w)$ must be bounded by $D''p^k$ for all p for some constant D'' ; since this intersection is a union of affine spaces, whose number of components is bounded by the number of pairs of components, we can choose one D'' which works for all w . Combining our estimates, we find:

1. The number of $w \in \widehat{W}$ with $\ell_0(w) \leq nm$, such that $\dim \overline{\text{Supp}(M^B)} \cap w \cdot \overline{\text{Supp}(M^B)} = k$ is bounded above by $D'm^{2d-k}$.
2. The dimension of $(A(\leq w) \cap A_0^m)/(A(< w) \cap A_0^m)$ is bounded above by $D''(Dm)^k$ if $\ell_0(w) \leq nm$.

Thus, summing over $k = 1, \dots, 2d$, we have $\dim A_0^m \leq D'D''D^{2d}m^{2d}$. Thus, we have

$$\log_m(\dim A_0^m) \leq 2d + \frac{\log(2dD'D'')D^{2d}}{\log m}$$

so taking the limit, we have $\text{GK-dim}(A) \leq 2d$, completing the proof. □

5. Functors from parabolic restriction

5.1. Restriction and induction functors

The relations of Theorem 2.14 induce a number of functors.

1. The Morita equivalence of Proposition 2.4 gives us an equivalence of categories

$$\mathcal{A}(G, N)\text{-mod} \cong \mathcal{A}^P(G, N)\text{-mod}.$$

This functor $\mathcal{A}^P(G, N)\text{-mod} \rightarrow \mathcal{A}(G, N)\text{-mod}$ can be written as $M \mapsto e'M$. Let $M^P = \mathcal{A}^P e' \otimes_{\mathcal{A}} M$ be the inverse equivalence.

2. Restriction induces a functor

$$\mathcal{A}^P(G, N)\text{-mod} \rightarrow \mathcal{A}(L, N)\text{-mod}.$$

This functor has a left adjoint given by $\mathcal{A}^P(G, N) \otimes_{\mathcal{A}(L, N)} -$ and a right adjoint $\text{Hom}_{\mathcal{A}(L, N)}(\mathcal{A}^P(G, N), -)$.

3. Tensor product $\mathcal{A}(L, N_0^\xi) \otimes_{\mathcal{A}(L, N)} -$ induces an exact functor

$$\mathcal{A}(L, N)\text{-mod} \rightarrow \mathcal{A}(L, N_0^\xi)\text{-mod}.$$

This has a right adjoint given by restriction.

Definition 5.1. We define the functor $\text{res}_\xi = \text{res}: \mathcal{A}(G, N)\text{-mod} \rightarrow \mathcal{A}(L, N_0^\xi)\text{-mod}$ as the composition of the above three functors

$$\mathcal{A}(G, N)\text{-mod} \rightarrow \mathcal{A}^P(G, N)\text{-mod} \rightarrow \mathcal{A}(L, N)\text{-mod} \rightarrow \mathcal{A}(L, N_0^\xi)\text{-mod}.$$

Lemma 5.2. The functor $\text{res}: \mathcal{A}(G, N)\text{-mod} \rightarrow \mathcal{A}(L, N_0^\xi)\text{-mod}$ is exact and admits a right adjoint $\text{coind}: \mathcal{A}(L, N_0^\xi)\text{-mod} \rightarrow \mathcal{A}(G, N)\text{-mod}$.

Proof. The right adjoint is just defined by composing the right adjoints of the functors (1–3) above. The Functor (1) above is exact since it is a Morita equivalence. The Functor (2) is exact since it is the same underlying vector space, and the Functor (3) is exact since it is a localization of the action of a polynomial ring. \square

5.2. Functors on Gelfand-Tsetlin modules

Lemma 5.3. *The functor res preserves Gelfand-Tsetlin modules.*

Proof. It’s clear that the Morita equivalence (1) preserves Gelfand-Tsetlin modules. Similarly, the restriction functor in (2) above clearly does so by compatibility with the Gelfand-Tsetlin subalgebras. The functor of tensor product with $\mathcal{A}(L, N_0^\xi) \otimes_{\mathcal{A}(L, N)} -$ preserves local finiteness under $\mathbb{A}_{\tilde{L}}$ since $\mathcal{A}(L, N_0^\xi)$ has the Harish-Chandra property as a bimodule over $\mathbb{A}_{\tilde{L}}$: the subbimodule generated by any finite set is finitely generated as a right module (and also as a left module, though that is not relevant here). For any element $v \in \mathcal{A}(L, N_0^\xi) \otimes_{\mathcal{A}(L, N)} M$, the subspace $\mathbb{A}_{\tilde{L}} \cdot v$ lies in the image of a tensor product $B \otimes_{\mathbb{A}_{\tilde{L}}} C$ of a finitely generated sub- $\mathbb{A}_{\tilde{L}} - \mathbb{A}_{\tilde{L}}$ -bimodule B and a finite-dimensional $\mathbb{A}_{\tilde{L}}$ -submodule C . Since B is finitely generated as a right module, $B \otimes_{\mathbb{A}_{\tilde{L}}} C$ is finite-dimensional, showing the local finiteness. \square

Let us examine a little more carefully the effect of res on weight spaces.

5.2.1. Morita equivalence on GT weight spaces

The effect of the Morita equivalence from Proposition 2.4 on weight spaces is covered in [Weba, Lemma 2.8]. Let $v \in \tilde{\mathfrak{t}}/W_L$, and γ its image in $\tilde{\mathfrak{t}}/W$. The preimage of γ in $\tilde{\mathfrak{t}}$ is a single orbit of the Weyl group W , and v corresponds to an orbit of W_L within this larger orbit. Generically, both these orbits are free, but there are degenerate cases where they are not. The effect of the Morita equivalence on weight spaces is a bit subtle in the latter case. Let $\lambda \in \tilde{\mathfrak{t}}$ be a preimage of v , let W^λ be its stabilizer in W and W_L^λ its stabilizer in W_L .

Lemma 5.4. *Let λ, v, γ correspond as above. Let M be an $\mathcal{A}(G, N)$ -module, and M^P the corresponding $\mathcal{A}^P(G, N)$ -module. If $W^\lambda = W_L^\lambda$, then ${}^P\mathcal{W}_v(M^P) = \mathcal{W}_\gamma(M)$; more generally, we have a functorial isomorphism ${}^P\mathcal{W}_v(M^P) \cong \mathcal{W}_\gamma(M)^{\oplus k}$, where $k = [W^\lambda : W_L^\lambda]$.*

Proof. Let $e'(\lambda)$ be the symmetrizing idempotent for W^λ and $e'_L(\lambda)$ be the symmetrizing idempotent for W_L^λ . Let M^B be the corresponding module over \mathcal{A}^B . By [Weba, Lemma 3.2], we have that

$$\mathcal{W}_\gamma(M) = e'(\lambda) \cdot {}^B\mathcal{W}_\lambda(M^B) \quad \mathcal{W}_v^P(M^P) = e'_L(\lambda) \cdot {}^B\mathcal{W}_\lambda(M^B)$$

and ${}^B\mathcal{W}_\lambda(M^B)$ is a free module over $\mathbb{C}[W^\lambda]$. Since $e'_L(\lambda)\mathbb{C}[W^\lambda] \cong \mathbb{C}^k$, the result follows. \square

5.2.2. Restriction on weight spaces

The inclusion of $\mathcal{A}(L, N)$ into $\mathcal{A}^P(G, N)$ from Proposition 2.5 identifies their Gelfand-Tsetlin subalgebras $\mathbb{A}_{\tilde{L}}$. In particular, restriction from $\mathcal{A}^P(G, N)$ to $\mathcal{A}(L, N)$ leaves weight spaces unchanged.

5.2.3. Inverting r_ξ and weight spaces

Finally, we wish to consider the effect of inverting r_ξ . This can be computed separately on the summands of any decomposition into $\mathbb{C}[r_\xi]$ -submodules. A Gelfand-Tsetlin module has a natural such decomposition, given by the sum of weight spaces in a single $\mathbb{Z}\xi$ -coset.

Let $v \in \tilde{\mathfrak{t}}/W_L$. Note that $v + \xi$ is well-defined since ξ is invariant under the action of W_L . Let M be a $\mathcal{A}(L, N)$ module. We consider the directed system

$$\dots \xrightarrow{r_\xi} \mathcal{W}_{v+\xi}^{L, N}(M) \xrightarrow{r_\xi} \mathcal{W}_v^{L, N}(M) \xrightarrow{r_\xi} \mathcal{W}_{v-\xi}^{L, N}(M) \xrightarrow{r_\xi} \dots \tag{5.1}$$

Let $\vec{W}_{[\nu]}^{L,N}(M)$ denote the direct limit $\varinjlim_{\nu-k\xi} \mathcal{W}_{\nu-k\xi}^{L,N}(M)$ of this system (here, $[\nu]$ denotes the image of ν in $(\tilde{\mathfrak{t}}/W_L)/\mathbb{Z}\xi$).

Lemma 5.5. *Let M be a Gelfand-Tsetlin $\mathcal{A}(L, N)$ -module. Then $\mathcal{A}(L, N_0^\xi) \otimes_{\mathcal{A}(L,N)} M$ is also a Gelfand-Tsetlin module, and for any $\nu \in \tilde{\mathfrak{t}}/W_L$, we have*

$$\mathcal{W}_\nu^{L,N_0^\xi}(\mathcal{A}(L, N_0^\xi) \otimes_{\mathcal{A}(L,N)} M) \cong \vec{W}_{[\nu]}^{L,N}(M).$$

In particular, this functor is exact on the category of Gelfand-Tsetlin modules.

Proof. Let $M' = \mathcal{A}(L, N_0^\xi) \otimes_{\mathcal{A}(L,N)} M$. We have an obvious map $p: M \rightarrow M'$ which leads to maps $r_\xi^{-k} p: \mathcal{W}_{\nu-k\xi}^{L,N}(M) \rightarrow \mathcal{W}_\nu^{L,N_0^\xi}(M')$, which are compatible with the directed system (5.1). This induces a map $\vec{W}_{[\nu]}^{L,N}(M) \rightarrow \mathcal{W}_\nu^{L,N_0^\xi}(M')$.

Since $\mathcal{A}(L, N_0^\xi) = \mathcal{A}(L, N)[r_\xi^{-1}]$, for any $\nu \in \mathcal{W}_\nu^{L,N_0^\xi}(M')$, we have that $r_\xi^k \nu$ is in the image of M for k sufficiently large. This shows that the map $\vec{W}_{[\nu]}^{L,N}(M) \rightarrow \mathcal{W}_\nu^{L,N_0^\xi}(M')$ is surjective.

On the other hand, if there is an element of the kernel, it is represented by some $w \in \mathcal{W}_{\nu-k\xi}^{L,N}(M)$ for some k . Since this element is killed by p , we have that $r_\xi^{k'} w = 0$ for some k' . Thus, w has trivial image in the directed limit. This shows injectivity. \square

However, the limit $\vec{W}_{[\nu]}^{L,N}(M)$ is already isomorphic to a weight space of M but possibly for a different ν . To see this, recall that in $\mathcal{A}(L, N)$, we have by (2.4) that

$$\begin{aligned} r_{-\xi} r_\xi &= \left(\prod_{\langle \mu, \xi \rangle > 0} \prod_{j=1}^{\langle \mu, \xi \rangle} (\mu - j) \right) \left(\prod_{\langle \mu, \xi \rangle < 0} \prod_{j=0}^{-\langle \mu, \xi \rangle - 1} (\mu + j) \right) \\ r_\xi r_{-\xi} &= \left(\prod_{\langle \mu, \xi \rangle < 0} \prod_{j=1}^{-\langle \mu, \xi \rangle} (\mu - j) \right) \left(\prod_{\langle \mu, \xi \rangle > 0} \prod_{j=0}^{\langle \mu, \xi \rangle - 1} (\mu + j) \right), \end{aligned} \tag{5.2}$$

where the products both range over subsets of the weights μ of the representation N , counted with multiplicity.

These formulas motivate the following definition.

Definition 5.6. $\lambda \in \tilde{\mathfrak{t}}$ is called ξ -negative, if

1. there is no weight μ of N , such that $\langle \mu, \xi \rangle > 0$ and $\langle \mu, \lambda \rangle$ is a positive integer and
2. there is no weight μ of N , such that $\langle \mu, \xi \rangle < 0$ and $\langle \mu, \lambda \rangle$ is a nonpositive integer.
3. The stabilizer W^λ lies in W_L , that is $W^\lambda = W_L^\lambda$.

Since ξ is invariant under W_L and the set of weights of N is invariant under W_L , the set of ξ -negative elements of $\tilde{\mathfrak{t}}$ is invariant under W_L . Thus, it makes sense to speak of ξ -negative elements of $\tilde{\mathfrak{t}}/W_L$.

Lemma 5.7. *Assume that $\nu \in \tilde{\mathfrak{t}}/W_L$ is ξ -negative.*

1. For all $k \in \mathbb{Z}_{\geq 0}$, r_ξ^k gives an isomorphism of functors

$$\mathcal{W}_{\nu-k\xi}^{L,N} \rightarrow \mathcal{W}_{\nu-(k+1)\xi}^{L,N}.$$

2. The natural map $\mathcal{W}_\nu^{L,N} \rightarrow \vec{W}_{[\nu]}^{L,N}$ is an isomorphism of functors.

Proof. Let M be a Gelfand-Tsetlin $\mathcal{A}(L, N)$ -module. Let $k \in \mathbb{Z}_{\geq 0}$. We consider the composition $r_{-\xi} r_{\xi}$ as a linear operator on $\mathcal{W}_{\nu-k\xi}(M)$. By (5.2), the eigenvalues of this operator on this weight space are given by the set

$$\begin{aligned} & \{ \langle \mu, \nu \rangle - k \langle \mu, \xi \rangle - j : \langle \mu, \xi \rangle > 0, 1 \leq j \leq \langle \mu, \xi \rangle \} \\ & \cup \{ \langle \mu, \nu \rangle - k \langle \mu, \xi \rangle + j : \langle \mu, \xi \rangle < 0, 0 \leq j < -\langle \mu, \xi \rangle \}. \end{aligned}$$

The hypothesis of ξ -negativity ensures that none of these eigenvalues vanish. Thus, $r_{-\xi} r_{\xi}$ is an isomorphism. Similarly, we see that $r_{\xi} r_{-\xi}$ is an isomorphism on $\mathcal{W}_{\nu-(k+1)\xi}(M)$.

So we conclude that r_{ξ} is an isomorphism. Then the second part follows immediately. □

5.2.4. Combined effect of res on weight spaces

Combining the results above, we conclude the following.

Theorem 5.8. *Consider a Gelfand-Tsetlin $\mathcal{A}(G, N)$ module M . Let $\nu \in \check{\mathfrak{t}}/W_L$, and let $\gamma \in \check{\mathfrak{t}}/W$ denote the image of ν . Assume that ν is ξ -negative. Then there is a natural isomorphism*

$$\mathcal{W}_{\nu}^L(res(M)) \cong \mathcal{W}_{\gamma}(M).$$

Corollary 5.9. *Let $\nu, \nu' \in \check{\mathfrak{t}}/W_L$ be ξ -negative, and let $\gamma, \gamma' \in \check{\mathfrak{t}}/W$ be their images. There is a morphism*

$$\text{Hom}(\mathcal{W}_{\nu}^L, \mathcal{W}_{\nu'}^L) \rightarrow \text{Hom}(\mathcal{W}_{\gamma}, \mathcal{W}_{\gamma'}).$$

Proof. By Theorem 5.8, we have

$$\mathcal{W}_{\nu}^L(res(M)) \cong \mathcal{W}_{\gamma}(M) \quad \mathcal{W}_{\nu'}^L(res(M)) \cong \mathcal{W}_{\gamma'}(M).$$

Given any $x \in \text{Hom}(\mathcal{W}_{\nu}^L, \mathcal{W}_{\nu'}^L)$, the map $\mathcal{W}_{\nu}^L(res(M)) \xrightarrow{x} \mathcal{W}_{\nu'}^L(res(M))$ gives us our desired map $\mathcal{W}_{\gamma}(M) \rightarrow \mathcal{W}_{\gamma'}(M)$. □

5.3. Algebraic description of the res functor

We will now combine Proposition 4.5 with Corollary 5.9 to obtain an algebraic description of the functor res .

Consider a \widehat{W} -orbit $\mathcal{S} \subset \check{\mathfrak{t}}$. As before, we let $\mathcal{A}(G, N) - \Gamma\Pi_{\mathcal{S}}$ be the category of Gelfand-Tsetlin modules supported on the image of this orbit in $\check{\mathfrak{t}}/W$. Since \mathcal{S} is closed under addition by ξ , Lemma 5.5 shows that if M is supported on \mathcal{S} , then $res(M)$ is supported on \mathcal{S} as well.

The set \mathcal{S} is a finite union of \widehat{W}^L -orbits. We will fix attention on a single one of these \widehat{W}^L -orbits, which we denote \mathcal{S}^L . Let $res_{\mathcal{S}^L}^{\xi} : \mathcal{A}(G, N) - \Gamma\Pi_{\mathcal{S}} \rightarrow \mathcal{A}(L, N_0^{\xi}) - \Gamma\Pi_{\mathcal{S}^L}$ be the functor given by applying res and then taking the summand supported on \mathcal{S}^L .

Let S, S^L be finite complete sets in $\mathcal{S}, \mathcal{S}^L$, respectively, and without loss of generality, assume that $S^L \subset S$.

Since r_{ξ} is invertible in $\mathcal{A}(L, N_0^{\xi})$ as in Theorem 2.9, we have $\mathcal{W}_{\nu}^{L, N_0^{\xi}} \cong \mathcal{W}_{\nu+\xi}^{L, N_0^{\xi}}$ for any $\nu \in \check{\mathfrak{t}}/W_L$. Thus, without loss of generality, we can choose all elements of S^L to be ξ -negative.

By Proposition 4.5, we have equivalences

$$\mathcal{A}(G, N) - \Gamma\Pi_{\mathcal{S}} \cong F(S) - \text{mod} \quad \mathcal{A}(L, N_0^{\xi}) - \Gamma\Pi_{\mathcal{S}^L} \cong F^L(S^L) - \text{mod}.$$

We can define natural $F^L(S^L), F(S)$ -bimodules corresponding to the restriction and induction functors:

$$I(S^L, S) = \bigoplus_{\substack{\gamma' \in \bar{S} \\ \nu \in \bar{S}^L}} \text{Hom}(\mathcal{W}_\nu^L \circ \text{res}, \mathcal{W}_{\gamma'}) \quad I(S, S^L) = \bigoplus_{\substack{\gamma' \in \bar{S} \\ \nu \in \bar{S}^L}} \text{Hom}(\mathcal{W}_{\gamma'}, \mathcal{W}_\nu^L \circ \text{res}), \quad (5.3)$$

where \bar{S} denotes the image of S in $\tilde{\mathfrak{t}}/W$ and \bar{S}^L denotes the image of S^L in $\tilde{\mathfrak{t}}/W_L$.

Recall that by construction, each $\nu \in S^L$ is ξ -negative. Thus, by Theorem 5.8, we can choose an isomorphism $\mathcal{W}_\nu^L \circ \text{res} \cong \mathcal{W}_\gamma$, where γ is the image of ν in $\tilde{\mathfrak{t}}/W$. This induces a vector space isomorphism $F(S, S^L) \cong I(S, S^L)$ (see (4.1) for the definition of $F(S, S^L)$). The right action of $F(S^L)$ on $F(S, S^L)$ and the right action of $F^L(S^L)$ on $I(S, S^L)$ are related by the homomorphism of Corollary 5.9.

Let $e^L \in F(S)$ be the idempotent obtained by summing the identities on elements of S^L . In this case, $e^L F(S) e^L = F(S^L)$ and $I(S, S^L) = e^L F(S)$. Note that for $M \in F(S)$ -mod, we have functorial isomorphisms

$$I(S, S^L) \otimes_{F(S)} M \cong e^L M \cong \text{Hom}_{F(S)}(I(S^L, S), M). \quad (5.4)$$

Theorem 5.10. *We have a commutative diagram*

$$\begin{array}{ccc} F(S)\text{-mod} & \xrightarrow{I(S, S^L) \otimes_{F(S)} -} & F^L(S^L)\text{-mod} \\ \mathcal{W}_S \uparrow & & \uparrow \mathcal{W}_{S^L}^L \\ \mathcal{A}(G, N) - \Gamma\Pi_{\mathcal{S}} & \xrightarrow{\text{res}} & \mathcal{A}(L, N_0^\xi) - \Gamma\Pi_{\mathcal{S}^L} \end{array}$$

Note that (5.4) allows us to construct left and right adjoints to res on the category of GT modules:

$$\begin{array}{ccc} F(S)\text{-mod} & \xleftarrow{I(S^L, S) \otimes_{F^L(S^L)} -} & F^L(S^L)\text{-mod} \\ \mathcal{W} \uparrow & & \uparrow \mathcal{W}^L \\ \mathcal{A}(G, N) - \Gamma\Pi_{\mathcal{S}} & \xleftarrow{\text{ind}} & \mathcal{A}(L, N_0^\xi) - \Gamma\Pi_{\mathcal{S}^L} \end{array}$$

$$\begin{array}{ccc} F(S)\text{-mod} & \xleftarrow{\text{Hom}_{F^L(S^L)}(I(S, S^L), -)} & F^L(S^L)\text{-mod} \\ \mathcal{W} \uparrow & & \uparrow \mathcal{W}^L \\ \mathcal{A}(G, N) - \Gamma\Pi_{\mathcal{S}} & \xleftarrow{\text{coind}} & \mathcal{A}(L, N_0^\xi) - \Gamma\Pi_{\mathcal{S}^L} \end{array}$$

The right adjoint coind is well-defined on all modules, based on the definition in Section 5.1. Note that it's not clear that ind is well-defined for all modules, though it seems likely that it agrees with coind for the coweight $-\xi$. We can also construct ind from coind by using duality on the category of Gelfand-Tsetlin modules.

5.4. Hamiltonian reduction

In this section, we consider Part (4) of Theorem 2.14, and the corresponding functors on modules induced by quantum Hamiltonian reduction.

Let A be an algebra and $b \in A$, and recall that $A //_1 b = \text{End}_A(A/(b-1)A)$. There is a natural $A //_1 b$, A -bimodule structure on $A/(b-1)A$. In particular, there is a natural functor $A\text{-mod} \rightarrow A //_1 b\text{-mod}$ defined by

$$M \mapsto A/(b-1)A \otimes_A M \cong M/(b-1)M,$$

which has right adjoint $\text{Hom}_{A //_1 b}(A/(b-1)A, -)$.

An instructive example to think about is when $G = \mathbb{C}^\times$ and the action on N is trivial (in this case, the algebra $\mathcal{A}(\mathbb{C}^\times, N)$ does not depend on N). By [BFN18, Section 4(iv)], the resulting algebra is generated by $x = r_{-1}, x^{-1} = r_1$ and a with the relation $[a, x] = x$, where a is the equivariant parameter coming from $\mathbb{A}_{\mathbb{C}^\times}$. We can either think of this algebra as a ring of difference operators on the polynomial ring $\mathbb{C}[a]$, with x acting by translation, or as differential operators on the torus \mathbb{C}^\times with coordinate x , with $a = x \frac{\partial}{\partial x}$ (the isomorphism relating these two realizations is called the *Mellin transform*). In this case, the quantum Hamiltonian reduction is simply $\mathcal{A}(\mathbb{C}^\times, N) //_1 x \cong \mathbb{C}$.

The representation theory of $\mathcal{A}(\mathbb{C}^\times, N)$ is not trivial, but it is simple. A Gelfand-Tsetlin module M over $\mathcal{A}(\mathbb{C}^\times, N)$ is one on which a acts locally finitely, and can equivalently be thought of as a D-module on \mathbb{C}^\times on which $a = x \frac{\partial}{\partial x}$ acts locally finitely; this implies that M corresponds to a regular local system. Thus, a simple module of this type must be a 1-dimensional local system with monodromy $c \in \mathbb{C}^\times$. This module is isomorphic to $\mathcal{A}/\mathcal{A}(a-m)$ for any m , such that $e^{2\pi im} = c$.

For a general module M , $M/(x-1)M$ is the fibre of the local system at $x = 1$. In particular, for a Gelfand-Tsetlin module, the number of simple composition factors is the dimension of this fibre. In order to reconstruct M , we need to also remember the action of the monodromy map $\exp(2\pi ia)$ on this quotient (which is well-defined).

Now, consider the situation of Theorem 2.11: Assume that $\mathbb{C}_\xi^\times \subset G$ is the image of a primitive cocharacter $\xi: \mathbb{C}^\times \rightarrow G$ which acts trivially on a representation N . Recall from Theorem 2.11 that, in this case, $\mathcal{A}(G, N) //_1 r_\xi \cong \mathcal{A}(G/\mathbb{C}_\xi^\times, N)$.

Let $\tilde{\mathfrak{t}}' = \tilde{\mathfrak{t}}/\mathbb{C}_\xi^\times$, where we use ξ to denote the derivative of the cocharacter ξ . That is, $\tilde{\mathfrak{t}}'$ is the Lie algebra of a maximal torus of $\tilde{G}' = \tilde{G}/\mathbb{C}_\xi^\times$. Thus, we have a map $\mathbb{C}[\tilde{\mathfrak{t}}']^W \subset \mathbb{C}[\tilde{\mathfrak{t}}]^W \rightarrow \mathcal{A}(G, N)$; the composed map commutes with r_ξ , so $\mathbb{C}[\tilde{\mathfrak{t}}']^W$ maps into the Hamiltonian reduction. This induces the usual map $\mathbb{A}_{G'} \cong \mathbb{C}[\tilde{\mathfrak{t}}']^W \rightarrow \mathcal{A}(G', N)$.

We have a Hamiltonian reduction functor on left modules:

$$M \mapsto M/(r_\xi - 1)M.$$

Proposition 5.11. *Let $M \in \mathcal{A}(G, N) - \Gamma\Pi$. Then $M/(r_\xi - 1)M$ is a Gelfand-Tsetlin module for $\mathcal{A}(G', N)$, and for $\gamma' \in \tilde{\mathfrak{t}}'/W$ we have*

$$\mathcal{W}_{\gamma'}(M/(r_\xi - 1)M) \cong \bigoplus_{[\gamma] \in (\gamma' + \mathbb{C}\xi)/\mathbb{Z}\xi} \mathcal{W}_\gamma(M),$$

where the direct sum ranges over a set of representatives modulo $\mathbb{Z}\xi$ for the preimage of γ' in $\tilde{\mathfrak{t}}/W$.

For γ a $\mathbb{Z}\xi$ -coset representative, the weight space $\mathcal{W}_\gamma(M)$ can be written more canonically as the limit of the directed system (5.1). Since \mathbb{C}_ξ^\times acts trivially on N , every map of this directed system will be an isomorphism.

Proof. For any $\gamma \in \mathfrak{t}/W$, the weight spaces $\mathcal{W}_\gamma(M)$ and $\mathcal{W}_{\gamma+k\xi}(M)$ are canonically isomorphic by $r_{\pm k\xi}$. Since all weight spaces of M are finite-dimensional, it follows that $\bigoplus_{k \in \mathbb{Z}} \mathcal{W}_{\gamma+k\xi}(M)$ is a free module of finite rank over the Laurent polynomial ring $\mathbb{C}[r_\xi, r_{-\xi}]$, which is freely generated by $\mathcal{W}_\gamma(M)$.

Since M is finitely generated, the support of M is a finite union of orbits of the extended affine Weyl group of G , and so there are only finitely many cosets of $\mathbb{Z}\xi$ in any given coset of $\mathbb{C}\xi$ that lie in the support of M .

Given $\gamma' \in \tilde{\mathfrak{t}}'/W$, we have that $M'_{\gamma'} := \bigoplus_{\gamma \in \gamma' + \mathbb{C}\xi} \mathcal{W}_\gamma(M)$ is also a free module of finite rank over $\mathbb{C}[r_\xi, r_{-\xi}]$: there are only finitely many cosets $[\gamma] \in (\gamma' + \mathbb{C}\xi)/\mathbb{Z}\xi$ for which $\mathcal{W}_\gamma(M) \neq 0$, and we may apply the above argument on each coset. We can choose free generators for $M'_{\gamma'}$ by fixing one representative γ from each such coset $[\gamma]$, and thus

$$M/(r_\xi - 1)M = \bigoplus_{\gamma'} M'_{\gamma'}/(r_\xi - 1)M'_{\gamma'} = \bigoplus_{\gamma'} \left(\bigoplus_{[\gamma] \in (\gamma' + \mathbb{C}\xi)/\mathbb{Z}\xi} \mathcal{W}_\gamma(M) \right). \quad \square$$

Since ξ is primitive, we can find $a \in \tilde{\mathfrak{t}}_{\mathbb{Z}}^*$ with $\xi(a) = 1$. Choose a logarithm map $\log : \mathbb{C}^\times \rightarrow \mathbb{C}$ inverse to $m \mapsto e^{2\pi im}$. For simplicity, we can uniquely fix this by requiring its image to lie in $[0, 1) + i\mathbb{R}$.

The adjoint action of a on $\mathcal{A}(G, N)$ integrates to a \mathbb{C}^\times action and thus has only integer eigenvalues. Thus, the module M is the direct sum of submodules

$$M_c := \bigoplus_{k \in \mathbb{Z}} \mathcal{W}_{\log c + k}^a(M), \tag{5.5}$$

where $\mathcal{W}_x^a(M)$ denotes the generalized x -eigenspace for the action of a on M , and where c ranges over \mathbb{C}^\times .

We use a to split $\tilde{\mathfrak{t}} = \tilde{\mathfrak{t}}' \oplus \mathbb{C}\xi$. For $c \in \mathbb{C}^\times$, define

$$(M/(r_\xi - 1)M)_c := \mathcal{W}_{\log c}^a(M) = \bigoplus_{\gamma' \in \tilde{\mathfrak{t}}'/W} \mathcal{W}_{\gamma' + (\log c)\xi}(M).$$

Via Proposition 5.11, we can identify $(M/(r_\xi - 1)M)_c$ with a subspace of $M/(r_\xi - 1)M$.

This gives a decomposition

$$M/(r_\xi - 1)M = \bigoplus_{c \in \mathbb{C}^\times} (M/(r_\xi - 1)M)_c = \bigoplus_{c \in \mathbb{C}^\times} M_c/(r_\xi - 1)M_c \tag{5.6}$$

into $\mathcal{A}(G', N)$ -submodules by reducing the decomposition (5.5).

To connect this to the D -modules on the punctured line described above, note that a, r_ξ^\pm generate a copy of $D(\mathbb{C}^\times)$ inside of $\mathcal{A}(G, N)$. The module M can be viewed as a D -module on \mathbb{C}^\times via this isomorphism. Then the left-hand side of (5.6) is the fibre of this D -module at 1 and the right-hand side is the decomposition of the fibre (or equivalently, the nearby cycles at the origin) into generalized eigenspaces for the monodromy around the origin.

6. Geometric description of weight modules and functors

6.1. Recollection on earlier results

By Proposition 4.5, spaces of natural transformations between weight functors control the category of Gelfand-Tsetlin modules. In the papers [Weba, Webb], the third author gave a geometric description of these spaces. We will now recall this description; it will be phrased as an equivalence of categories. We begin by defining the two categories involved.

In this section, we will only work with integral weights. Here, $\tilde{\mathfrak{t}}_{\mathbb{Z}} = \text{Hom}(\mathbb{C}^\times, \tilde{T}) \subset \tilde{\mathfrak{t}}$.

6.1.1. Category of weight functors

Definition 6.1. We call a Gelfand-Tsetlin module over $\mathcal{A}(G, N)$ or $\mathcal{A}^B(G, N)$ **integral** if its support lies in $\tilde{\mathfrak{t}}_{\mathbb{Z}}$.

Let $\widehat{\mathcal{A}}_{\mathbb{Z}}(G, N)$ be the category with objects $\tilde{t}_{\mathbb{Z}}/W$ and morphisms given by

$$\text{Hom}_{\widehat{\mathcal{A}}_{\mathbb{Z}}(G, N)}(\gamma, \gamma') = \text{Hom}(\mathcal{W}_{\gamma}, \mathcal{W}_{\gamma'}).$$

Similarly, let $\widehat{\mathcal{A}}_{\mathbb{Z}}^B(G, N)$ be the category with objects $\tilde{t}_{\mathbb{Z}}$ and morphisms given by

$$\text{Hom}_{\widehat{\mathcal{A}}_{\mathbb{Z}}^B(G, N)}(\lambda, \lambda') = \text{Hom}({}^B\mathcal{W}_{\lambda}, {}^B\mathcal{W}_{\lambda'}),$$

where, as before, ${}^B\mathcal{W}_{\lambda}$ is a weight functor on the category of $\mathcal{A}^B(G, N)$ -modules.

Remark 6.2. The reader might naturally wonder about the relationship between $\widehat{\mathcal{A}}_{\mathbb{Z}}(G, N)$ and $\widehat{\mathcal{A}}_{\mathbb{Z}}^B(G, N)$. They are not equivalent: for example, in the pure case where $N = 0$, if γ is a W -orbit with a single element, the endomorphism algebra of γ in $\widehat{\mathcal{A}}(G, N)$ has a single 1-dimensional discrete irreducible representation, and no object in $\widehat{\mathcal{A}}^B(G, N)$ has this property. On the other hand, by Lemma 5.4, there is an equivalence between the Karoubian envelopes of these categories, sending $\lambda \in \tilde{t}_{\mathbb{Z}}$ to the direct sum of W^{λ} copies of its image in $\tilde{t}_{\mathbb{Z}}/W$. The inverse functor thus sends an element of $\tilde{t}_{\mathbb{Z}}/W$ corresponding to a nonfree orbit to the image of the symmetrizing idempotent $e'(\lambda)$ in W^{λ} acting on a preimage λ .

6.1.2. Steinberg category

Next, we will define certain Steinberg-type varieties, which will be the building blocks of our second category.

First, given coweights $\gamma, \gamma' \in \tilde{t}_{\mathbb{Z}}/W$, as before, let λ, λ' be the antidominant lifts of γ, γ' . As above, N_{\leq}^{λ} is the subspace in N on which λ has nonpositive weight, and let $P_{\lambda} \subset G$ be the parabolic subgroup on whose Lie algebra λ has nonpositive weight (and similarly for λ'). Let:

$$Y_{\gamma} = (G \times N_{\leq}^{\lambda})/P_{\lambda} \cong \{(gP_{\lambda}, n) \mid n \in gN_{\leq}^{\lambda}\} \tag{6.1}$$

$${}_{\gamma}\mathbb{Y}_{\gamma'} = Y_{\gamma} \times_N Y_{\gamma'} = \{(g_1P_{\lambda}, g_2P_{\lambda'}, n) \mid n \in g_1N_{\leq}^{\lambda} \cap g_2N_{\leq}^{\lambda'}\}. \tag{6.2}$$

Let $\widehat{H}^G({}_{\gamma}\mathbb{Y}_{\gamma'})$ denote the completion of the equivariant Borel-Moore homology of ${}_{\gamma}\mathbb{Y}_{\gamma'}$ with respect to its grading.

We will also need a full flag version of this construction. Let:

$$X_{\lambda} = (G \times N_{\leq}^{\lambda})/B \quad {}_{\lambda}\mathbb{X}_{\lambda'} = X_{\lambda} \times_N X_{\lambda'} = \{(g_1B, g_2B, n) \mid n \in g_1N_{\leq}^{\lambda} \cap g_2N_{\leq}^{\lambda'}\}. \tag{6.3}$$

Note that $B \subset P_{\lambda}$ and if λ is the antidominant lift of γ , we have a natural morphism $X_{\lambda} \rightarrow Y_{\gamma}$ which is a P_{λ}/B bundle.

We can easily extend the definition of these spaces when λ, λ' are not antidominant. To this end, we define B_{λ} to be the Borel subgroup whose Lie algebra consists of those root spaces \mathfrak{g}_{α} , for all α , such that either $\langle \lambda, \alpha \rangle < 0$, or $\langle \lambda, \alpha \rangle = 0$ and α is negative. Then N_{\leq}^{λ} will be invariant under B_{λ} and we define $X_{\lambda} = (G \times N_{\leq}^{\lambda})/B_{\lambda}$. Note that this space would be the same for any other Borel contained in P_{λ} ; any two such Borels are conjugate in P_{λ} , and any element in P_{λ} conjugating between them induces an isomorphism by $(g, n) \mapsto (gp^{-1}, pn)$. More generally, this assignment of spaces to coweights is equivariant for the action of the Weyl group. Even though $wB_{\lambda}w^{-1} \neq B_{w\lambda}$ in some cases, we still have $wB_{\lambda}w^{-1} \subset P_{w\lambda}$, and so $X_{\lambda} \cong X_{w\lambda}$. Finally, we extend the definition (6.3) of ${}_{\lambda}\mathbb{X}_{\lambda'}$ to the case of arbitrary $\lambda, \lambda' \in \tilde{t}_{\mathbb{Z}}$.

Let $\widehat{H}^G({}_{\lambda}\mathbb{X}_{\lambda'})$ denote the completion of the equivariant Borel-Moore homology of ${}_{\lambda}\mathbb{X}_{\lambda'}$ with respect to its grading.

Remark 6.3. When there is an ambiguity, we will write $Y_{\gamma}^{G, N}$ etc. to keep track of the group G and representation N .

Definition 6.4. Let $\widehat{\mathcal{X}}(G, N)$ be the category with objects $\widehat{\mathfrak{t}}_{\mathbb{Z}}/W$ and morphisms

$$\text{Hom}_{\widehat{\mathcal{X}}(G, N)}(\gamma, \gamma') = \widehat{H}^G(\gamma \mathbb{Y}_{\gamma'}).$$

Similarly, let $\widehat{\mathcal{X}}^B(G, N)$ be the category with objects $\widehat{\mathfrak{t}}_{\mathbb{Z}}$ and morphisms

$$\text{Hom}_{\widehat{\mathcal{X}}^B(G, N)}(\lambda, \lambda') = \widehat{H}^G(\lambda \mathbb{X}_{\lambda'}).$$

Composition in these categories is defined by convolution, as defined in [CG97, (2.7.9)], viewing $\lambda \mathbb{X}_{\lambda'} \subset X_{\lambda} \times X_{\lambda'}$ (in the notation of [CG97], $M_1 = X_{\lambda}$, $M_2 = X_{\lambda'}$). The associativity of the convolution product is given by [CG97, Section 2.7.18].

The category $\widehat{\mathcal{X}}^B(G, N)$ is a variation of the Steinberg category, defined in [Webb, Section 2.4]. The definition from [Webb] involves assigning a subspace in N to each element of $\widehat{\mathfrak{t}}_{\mathbb{Z}}$, which we have done here by the assignment $\lambda \mapsto N_{\leq}^{\lambda}$; in [Webb, Section 2.4], this subspace is encoded as a sign sequence on a basis of N representing which basis vectors lie in N_{\leq}^{λ} and which do not.

6.1.3. The equivalences

Following [Webb, Definition 4.2], we will now construct equivalences between the weight functor categories and the Steinberg categories. Our strategy will be to begin with $\widehat{\mathcal{X}}(T, N)$, $\widehat{\mathcal{A}}_{\mathbb{Z}}(T, N)$, then study $\widehat{\mathcal{X}}^B(G, N)$, $\widehat{\mathcal{A}}_{\mathbb{Z}}^B(G, N)$ and finally $\widehat{\mathcal{X}}(G, N)$, $\widehat{\mathcal{A}}_{\mathbb{Z}}(G, N)$.

Recall from Section 2.3, that the algebra $\mathcal{A}_{\varphi}(T, N)$ is generated over \mathbb{A}_T by the elements r_{ν} (for $\nu \in \mathfrak{t}_{\mathbb{Z}}$), with relations given by (2.4).

For $\lambda, \lambda' \in \mathfrak{t}_{\mathbb{Z}}$, we define $\Phi_0(\lambda, \lambda') \in \mathbb{A}_T$ by the following formula, where the product ranges over the weights μ of the representation N , counted with multiplicity:

$$\Phi_0(\lambda, \lambda') = \prod_{\mu} \prod_{\substack{j=1, \dots, -\langle \mu, \lambda - \lambda' \rangle \\ j \neq \langle \mu, \lambda' \rangle}} (\mu - j). \tag{6.4}$$

Note that the polynomial $\Phi_0(\lambda, \lambda')$ acts invertibly on the functor \mathcal{W}_{λ} , so we can define morphisms in $\widehat{\mathcal{A}}_{\mathbb{Z}}(T, N)$ by

$$\mathfrak{w}(\lambda, \lambda') = \frac{1}{\Phi_0(\lambda, \lambda')} r_{\lambda - \lambda'} : \mathcal{W}_{\lambda} \rightarrow \mathcal{W}_{\lambda'}.$$

Note that ${}_{\lambda} \mathbb{X}_{\lambda}^{T, N}$ is a vector space. We also let $\mathfrak{w}(\lambda, \lambda')$ denote its fundamental class $[{}_{\lambda} \mathbb{X}_{\lambda}^{T, N}]$, which is a morphism in $\widehat{\mathcal{X}}(T, N)$.

A special case of [Webb, Theorem 4.3] is that:

Theorem 6.5. *There is an equivalence $E : \widehat{\mathcal{X}}(T, N) \cong \widehat{\mathcal{A}}_{\mathbb{Z}}(T, N)$ which is the identity on objects. On morphisms, this functor sends an element of $\mathfrak{t}^* \subset \mathbb{A}_T$ to the nilpotent part of the action of the same element in $\widehat{\mathcal{A}}_{\mathbb{Z}}(T, N)$, and takes $\mathfrak{w}(\lambda, \lambda')$ to the same-named morphism.*

From Proposition 2.5, we have an inclusion $\mathcal{A}_{\varphi}(T, N) \rightarrow \mathcal{A}_{\varphi}^B(G, N)$. This leads to a functor $\widehat{\mathcal{A}}_{\mathbb{Z}}(T, N) \rightarrow \widehat{\mathcal{A}}_{\mathbb{Z}}^B(G, N)$ which is the identity on objects. Describing the additional generators needed to generate $\mathcal{A}_{\varphi}^B(G, N)$ is tricky when working purely with the algebra \mathcal{A}_{φ} ; this process is simplified by considering the extended category introduced in [Webb, Section 3]. Here, we proceed a little differently.

Let \mathbb{L} denote the fraction field of \mathbb{A}_T and $\widehat{\mathbb{L}}$ the fraction field of the completion $\widehat{\mathbb{A}}_T$. Recall from [Webb, Proposition 4.2] that $\mathcal{A}_{\varphi}(T, N)$ is a principal Galois order inside the skew group algebra $\mathbb{L} \rtimes \mathfrak{t}_{\mathbb{Z}}$ for the usual action of $\mathfrak{t}_{\mathbb{Z}}$ on \mathbb{L} by translations on \mathfrak{t} . Similarly, $\mathcal{A}_{\varphi}(G, N)$ is a principal Galois order

inside the Weyl invariants of $(\mathbb{L} \rtimes \mathfrak{t}_{\mathbb{Z}})^W$. The algebra $\mathcal{A}_{\varphi}^B(G, N)$ is the corresponding flag order in the skew group algebra $\mathbb{L} \rtimes \widehat{W}$. That is, in particular, these inclusions induce isomorphisms:

$$\mathbb{L} \otimes_{\widehat{\Lambda}_T} \mathcal{A}_{\varphi}(T, N) \cong \mathbb{L} \rtimes \mathfrak{t}_{\mathbb{Z}}, \quad \mathbb{L}^W \otimes_{\widehat{\Lambda}_G} \mathcal{A}_{\varphi}(G, N) \cong (\mathbb{L} \rtimes \mathfrak{t}_{\mathbb{Z}})^W, \quad \mathbb{L} \otimes_{\widehat{\Lambda}_T} \mathcal{A}_{\varphi}^B(G, N) \cong \mathbb{L} \rtimes \widehat{W}.$$

By [Weba, Lemma 2.11(4)], the Hom space

$$\text{Hom}_{\widehat{\mathcal{A}}_{\mathbb{Z}}^B(G, N)}(\nu, \nu') = \mathcal{A}_{\varphi}^B / (\mathcal{A}_{\varphi}^B \mathfrak{m}_{\nu}^N + \mathfrak{m}_{\nu'}^N \mathcal{A}_{\varphi}^B),$$

is isomorphic to

$$\widehat{\Lambda}_T \otimes_{\widehat{\Lambda}_T} \left\{ \sum_{w \in \widehat{W}} a_w w \in \mathcal{A}_{\varphi}^B(G, N) \subset \mathbb{L} \rtimes \widehat{W} \mid a_w = 0 \text{ if } w\nu \neq \nu' \right\}. \tag{6.5}$$

Applying the same result with $G = B = T$ and the product decomposition $\widehat{W} = \mathfrak{t}_{\mathbb{Z}} \cdot W$, we obtain an isomorphism:

$$\widehat{\mathbb{L}} \otimes_{\widehat{\Lambda}_T} \text{Hom}_{\widehat{\mathcal{A}}_{\mathbb{Z}}^B(G, N)}(\nu, \nu') \cong \bigoplus_{w \in W} \widehat{\mathbb{L}} \otimes_{\widehat{\Lambda}_T} \text{Hom}_{\widehat{\mathcal{A}}_{\mathbb{Z}}(T, N)}(w\nu, \nu') \cdot w. \tag{6.6}$$

On the other hand, we also have a functor $\widehat{\mathcal{X}}(T, N) \rightarrow \widehat{\mathcal{X}}^B(G, N)$, which is the identity on objects and which acts on morphisms by

$$\widehat{H}^T(\lambda \mathbb{X}_{\lambda'}^{T, N}) \rightarrow \widehat{H}^T(\lambda \mathbb{X}_{\lambda'}^{G, N}) = \widehat{H}^B(\lambda \mathbb{X}_{\lambda'}^{G, N}) \rightarrow \widehat{H}^G(\lambda \mathbb{X}_{\lambda'}^{G, N}), \tag{6.7}$$

the composition of pushforward in T -equivariant homology, followed by G -saturation, the map $\widehat{H}^B(X) \cong \widehat{H}^G((G \times X)/B) \rightarrow \widehat{H}^G(X)$ which ‘averages’ B -equivariant cycles under G .

We can write

$$\begin{aligned} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}^B(G, N)}(\nu, \nu') &\cong \widehat{H}^B(\{(gB, n) \mid n \in N_{\leq}^{\lambda} \cap gN_{\leq}^{\lambda'}\}) \\ &\cong \widehat{H}^T(\{(gB, n) \mid n \in N_{\leq}^{\lambda} \cap gN_{\leq}^{\lambda'}\}). \end{aligned}$$

The fixed points of T on the space are given by $\{(wB, n) \mid w \in W, n \in N^T\}$. Applying localization in T -equivariant Borel-Moore homology, we find that we have a natural isomorphism:

$$\widehat{\mathbb{L}} \otimes_{\widehat{\Lambda}_T} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}^B(G, N)}(\nu, \nu') \cong \bigoplus_{w \in W} \widehat{\mathbb{L}} \otimes_{\widehat{\Lambda}_T} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(T, N)}(w\nu, \nu') \cdot w. \tag{6.8}$$

By [Webb, Theorem 4.3], we have the following result.

Theorem 6.6. *There is an equivalence $E: \widehat{\mathcal{X}}^B(G, N) \rightarrow \widehat{\mathcal{A}}_{\mathbb{Z}}^B(G, N)$ compatible with the isomorphisms (6.6) and (6.8), making the following diagram commute*

$$\begin{array}{ccc} \widehat{\mathcal{X}}(T, N) & \longrightarrow & \widehat{\mathcal{A}}_{\mathbb{Z}}(T, N) \\ \downarrow & & \downarrow \\ \widehat{\mathcal{X}}^B(G, N) & \longrightarrow & \widehat{\mathcal{A}}_{\mathbb{Z}}^B(G, N). \end{array}$$

Let λ, λ' be the antidominant lifts of $\gamma, \gamma' \in \check{\mathfrak{t}}_{\mathbb{Z}}/W$. The P_{λ}/B fibre bundle $X_{\lambda} \rightarrow Y_{\gamma}$ implies that the convolution algebra $H^G(X_{\lambda} \times_{Y_{\gamma}} X_{\lambda})$ is a copy of the nilHecke algebra for W^{λ} . By [CG97, Theorem 8.6.7], this algebra acts on the pushforward of the constant sheaf from X_{λ} . Since the nilHecke algebra is a matrix algebra on the commutative algebra $H_G^*(pt)$, this shows that this pushforward is a sum of

W copies of the constant sheaf on Y_γ . Thus, any primitive idempotent in this nilHecke algebra (in particular, the symmetrizing idempotent $e'(\lambda) \in \mathbb{C}W^\lambda$) gives this constant sheaf. This shows that:

$$\widehat{H}^L(\gamma \mathbb{Y}_{\gamma'}) \cong e'(\lambda) \widehat{H}^L(\lambda \mathbb{X}_{\lambda'}) e'(\lambda'). \tag{6.9}$$

By [Weba, Lemma 2.8] (also discussed in the proof of Lemma 5.4), we have a similar formula

$$\text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'}) \cong e'(\lambda) \text{Hom}({}^B\mathcal{W}_\lambda, {}^B\mathcal{W}_{\lambda'}) e'(\lambda'). \tag{6.10}$$

Thus, comparing equations (6.9) and (6.10), we have that:

Corollary 6.7. *There is an equivalence $E: \widehat{\mathcal{X}}(G, N) \rightarrow \widehat{\mathcal{A}}_{\mathbb{Z}}(G, N)$.*

Remark 6.8. In [Weba, proof of Theorem 4.4], following a suggestion of Nakajima, the third author gave a sketch proof of Theorem 6.6 and Corollary 6.7 using Abelianization in equivariant homology.

Remark 6.9. The effect of requiring Z -semisimplicity is encapsulated very cleanly in this theorem. If we consider the weight functors \mathcal{W}_γ^f on the category of all modules (rather than restricting to Z -semisimple ones), we obtain a different category, temporarily denoted $\widehat{\mathcal{A}}_{\mathbb{Z}}(G, N)^f$, whose morphism spaces are given by

$$\text{Hom}(\mathcal{W}_\gamma^f, \mathcal{W}_{\gamma'}^f) = \varprojlim \mathcal{A} / (\mathcal{A}m_\gamma^N + m_{\gamma'}^N \mathcal{A}).$$

A simple Gelfand-Tsetlin module over \mathcal{A} will factor through one of the quotients \mathcal{A}_φ , but there can be extensions of such modules where the centre Z acts nonsemisimply, and thus don't factor through any such quotient.

In order to fix Corollary 6.7 to work in this setting, we define $\widehat{\mathcal{X}}(G, N)^f$ by working with \widetilde{G} -equivariant cohomology instead of G -equivariant so that

$$\text{Hom}_{\widehat{\mathcal{X}}(G, N)^f}(\gamma, \gamma') = \widehat{H}^{\widetilde{G}}(\gamma \mathbb{Y}_{\gamma'}).$$

With these definitions, essentially the same argument gives us an equivalence of categories

$$\widehat{\mathcal{X}}(G, N)^f \rightarrow \widehat{\mathcal{A}}_{\mathbb{Z}}(G, N)^f.$$

The additional equivariant parameters for the larger group \widetilde{G} capture the action of the nilpotent part of Z , which as we mentioned above might be nontrivial.

6.2. Geometric viewpoint on parabolic restriction

Now, let us consider how to understand Corollary 5.9 in the context of the geometric description provided by Corollary 6.7.

Let $\nu, \nu' \in \check{\mathfrak{t}}_{\mathbb{Z}}/W_L$ be ξ -negative, and let $\gamma, \gamma' \in \check{\mathfrak{t}}_{\mathbb{Z}}/W$ denote their images. From Corollary 6.7, we have an isomorphism

$$\widehat{H}^G(\gamma \mathbb{Y}_{\gamma'}) \xrightarrow{\sim} \text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'}).$$

On the other hand, from Corollary 5.9, we have a morphism

$$\text{Hom}(\mathcal{W}_\nu^L, \mathcal{W}_{\nu'}^L) \rightarrow \text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'}),$$

where $\mathcal{W}_\nu^L, \mathcal{W}_{\nu'}^L$ denote the weight functors on the category of $\mathcal{A}(L, N_0^\xi)$ -modules.

We begin with the following observation, which follows immediately from Definition 5.6.

Lemma 6.10. *If ν is ξ -negative, then $Y_\nu^{L,N} = Y_\nu^{L,N_0^\xi} \times N_+^\xi$. This induces an isomorphism ${}_\gamma \mathbb{Y}_{\gamma'}^{L,N} \cong {}_\gamma \mathbb{Y}_{\gamma'}^{L,N_0^\xi} \times N_+^\xi$.*

Now, the map $L \rightarrow G$ induces a map $Y_\nu^L \rightarrow Y_\gamma$ and thus we have a map

$$\widehat{H}^L({}_\nu \mathbb{Y}_{\nu'}^L) \rightarrow \widehat{H}^P({}_\gamma \mathbb{Y}_{\gamma'}) \rightarrow \widehat{H}^G({}_\gamma \mathbb{Y}_{\gamma'}) \tag{6.11}$$

via G -saturation of P -equivariant cycles after pushforward as in (6.7); note that any parabolic P with Levi L will give the same result.

Equivalently, we can use the identifications

$$\begin{aligned} \widehat{H}^G({}_\gamma \mathbb{Y}_{\gamma'}) &\cong \widehat{H}^{P_\lambda}(\{(gP_\lambda, n) \mid n \in N_\leq^\lambda \cap gN_\leq^{\lambda'}\}) \\ \widehat{H}^L({}_\nu \mathbb{Y}_{\nu'}^L) &\cong \widehat{H}^{P_\lambda^L}(\{(gP_\lambda^L, n) \mid n \in N_\leq^\lambda \cap gN_\leq^{\lambda'}\}), \end{aligned}$$

where λ, λ' denote the antidominant lifts of γ, γ' . Since ν is ξ -negative, $W^\lambda \subset W_L$, and so P_λ^L and P_λ have the same reductive quotients. Thus, the map (6.11) is simply the pushforward in homology for the map

$$\{(gP_\lambda, n) \mid n \in N_\leq^\lambda \cap gN_\leq^{\lambda'}\} \rightarrow \{(gP_\lambda^L, n) \mid n \in N_\leq^\lambda \cap gN_\leq^{\lambda'}\}.$$

The functor E is far from the only equivalence between the corresponding categories. In order to compare the restriction functors with geometry, it is convenient to use a variation of the functor $E_L: \widehat{\mathcal{X}}(L, N_0^\xi) \rightarrow \widehat{\mathcal{A}}_{\mathbb{Z}}(L, N_0^\xi)$.

For each $\lambda \in \check{\mathfrak{t}}_{\mathbb{Z}}$, we define $\kappa_\lambda \in \mathbb{L}$ by the following product over the weights μ of N , taken with multiplicity:

$$\kappa_\lambda = \frac{\prod_{\substack{\langle \mu, \xi \rangle < 0 \\ \langle \mu, \lambda \rangle > 0}} \prod_{j=1}^{\langle \mu, \lambda \rangle - 1} (\mu - j)}{\prod_{\substack{\langle \mu, \xi \rangle < 0 \\ \langle \mu, \lambda \rangle \leq 0}} \prod_{j=0}^{-\langle \mu, \lambda \rangle - 1} (\mu + j)}. \tag{6.12}$$

Note that κ_λ acts invertibly on \mathcal{W}_λ^B ; since this assignment is W_L -equivariant, this induces a natural transformation on the functor \mathcal{W}_ν where ν is the image of λ in $\check{\mathfrak{t}}/W_L$. Let

$$E'_L = \kappa_{\nu'}^{-1} E_L \kappa_\nu: \widehat{H}^L({}_\nu \mathbb{Y}_{\nu'}^L) \rightarrow \text{Hom}(\mathcal{W}_\nu^L, \mathcal{W}_{\nu'}^L)$$

be the twist of the functor E_L from Corollary 6.7 by the maps κ_ν .

Lemma 6.11. *Assume that L is Abelian. The functor E'_L sends*

$$\mathbb{w}(\nu, \nu') \mapsto \frac{1}{\Phi'_0(\nu, \nu')} r_{\nu-\nu'}, \quad \Phi'_0(\nu, \nu') := \Phi_0^{L, N_0^\xi}(\nu, \nu') \prod_{\langle \mu, \xi \rangle < 0} \frac{\prod_{\substack{j=1, \dots, -\langle \mu, \nu-\nu' \rangle \\ j \neq \langle \mu, \nu' \rangle}} (\mu - j)}{\prod_{\substack{j=0, \dots, \langle \mu, \nu-\nu' \rangle - 1 \\ j \neq -\langle \mu, \nu' \rangle}} (\mu + j)}. \tag{6.13}$$

Proof. By definition, this functor sends

$$\begin{aligned} \mathbb{w}(\nu, \nu') &\mapsto \kappa_{\nu'}^{-1} \frac{1}{\Phi_0^{L, N_0^\xi}(\nu, \nu')} r_{\nu-\nu', \kappa_\nu} \\ &= \frac{1}{\Phi_0^{L, N_0^\xi}(\nu, \nu')} \frac{\prod_{\substack{\langle \mu, \xi \rangle < 0 \\ \langle \mu, \nu' \rangle \leq 0}} \prod_{j=0}^{-\langle \mu, \nu' \rangle - 1} (\mu + j) \prod_{\substack{\langle \mu, \xi \rangle < 0 \\ \langle \mu, \nu \rangle > 0}} \prod_{j=1+\langle \mu, \nu' - \nu \rangle}^{\langle \mu, \nu' \rangle - 1} (\mu - j)}{\prod_{\substack{\langle \mu, \xi \rangle < 0 \\ \langle \mu, \nu' \rangle > 0}} \prod_{j=1}^{\langle \mu, \nu' \rangle - 1} (\mu - j) \prod_{\substack{\langle \mu, \xi \rangle < 0 \\ \langle \mu, \nu \rangle \leq 0}} \prod_{j=\langle \mu, \nu - \nu' \rangle}^{-\langle \mu, \nu' \rangle - 1} (\mu + j)} r_{\nu-\nu'}. \end{aligned}$$

For any weight μ , its contribution to this product depends on the signs of $\langle \mu, \nu \rangle$ and $\langle \mu, \nu' \rangle$.

If $\langle \mu, \nu \rangle > 0, \langle \mu, \nu' \rangle \leq 0$, then the contribution of μ to the denominator is trivial, and its contribution to the numerator becomes

$$\prod_{j=0}^{-\langle \mu, \nu' \rangle - 1} (\mu + j) \prod_{j=-\langle \mu, \nu' \rangle + 1}^{-\langle \mu, \nu' - \nu \rangle - 1} (\mu + j) = \prod_{\substack{j=0, \dots, \langle \mu, \nu - \nu' \rangle - 1 \\ j \neq -\langle \mu, \nu' \rangle}} (\mu + j).$$

Dually, if $\langle \mu, \nu \rangle \leq 0, \langle \mu, \nu' \rangle > 0$, the numerator is trivial, and the denominator is

$$\prod_{j=1}^{\langle \mu, \nu' \rangle - 1} (\mu - j) \prod_{j=\langle \mu, \nu' \rangle + 1}^{\langle \mu, \nu' - \nu \rangle} (\mu - j) = \prod_{\substack{j=1, \dots, -\langle \mu, \nu - \nu' \rangle \\ j \neq \langle \mu, \nu' \rangle}} (\mu - j).$$

On the other hand, if $\langle \mu, \nu \rangle > 0, \langle \mu, \nu' \rangle > 0$, respectively, $\langle \mu, \nu \rangle \leq 0, \langle \mu, \nu' \rangle \leq 0$, then these contributions are:

$$\frac{\prod_{j=1+\langle \mu, \nu' - \nu \rangle}^{\langle \mu, \nu' \rangle - 1} (\mu - j)}{\prod_{j=1}^{\langle \mu, \nu' \rangle - 1} (\mu - j)}, \quad \text{respectively} \quad \frac{\prod_{j=0}^{-\langle \mu, \nu' \rangle - 1} (\mu + j)}{\prod_{j=\langle \mu, \nu - \nu' \rangle}^{-\langle \mu, \nu' \rangle - 1} (\mu + j)}.$$

Applying the obvious cancellation gives the desired result. □

Recall that in Theorem 5.8, we fixed a natural isomorphism of functors $\Upsilon_\nu : \mathcal{W}_\nu^L \circ \text{res} \rightarrow \mathcal{W}_\nu$. Also recall the map $\Upsilon : \text{Hom}(\mathcal{W}_\nu^L, \mathcal{W}_{\nu'}^L) \rightarrow \text{Hom}(\mathcal{W}_\nu, \mathcal{W}_{\nu'})$ from Corollary 5.9. By definition, for $x \in \text{Hom}(\mathcal{W}_\nu^L, \mathcal{W}_{\nu'}^L)$, we have $\Upsilon(x) = \Upsilon_{\nu'} x \Upsilon_\nu^{-1}$.

We will use these factors κ_ν to twist these maps, and define

$$\Upsilon'_\nu = \Upsilon_\nu \kappa_\nu^{-1} \quad \text{and} \quad \Upsilon'(x) = \Upsilon'_{\nu'} x (\Upsilon'_\nu)^{-1}. \tag{6.14}$$

Theorem 6.12. *Assume that $\nu, \nu' \in \tilde{\mathfrak{t}}_{\mathbb{Z}}/W_L$ are both ξ -negative. Let γ, γ' be their images in $\tilde{\mathfrak{t}}_{\mathbb{Z}}/W$. Then we have a commutative diagram*

$$\begin{array}{ccc}
 \widehat{H}^L(\nu \mathbb{Y}_{\nu'}) & \xrightarrow{(6.11)} & \widehat{H}^G(\gamma \mathbb{Y}_{\gamma'}) \\
 \downarrow E_L & & \downarrow E_G \\
 \text{Hom}(\mathcal{W}_{\nu}^L, \mathcal{W}_{\nu'}^L) & \xrightarrow{\Upsilon} & \text{Hom}(\mathcal{W}_{\gamma}, \mathcal{W}_{\gamma'}).
 \end{array}$$

Here, \mathcal{W}_{ν}^L denotes the weight functor on the category of $\mathcal{A}(L, N_0^{\xi})$ modules and \mathcal{W}_{γ} denotes the weight functor on the category of $\mathcal{A}(G, N)$ modules.

We'll complete this proof in a couple of steps. First, we consider the Abelian case:

Lemma 6.13. *Theorem 6.12 holds in the case where $G = T$ is Abelian.*

Proof. Note that, in this case, $L = T$ as well, and $\nu = \gamma, \nu' = \gamma'$. We can equivalently show the commutativity of the diagram:

$$\begin{array}{ccc}
 \widehat{H}^T(\nu \mathbb{Y}_{\nu'}) & \xrightarrow{(6.11)} & \widehat{H}^T(\nu \mathbb{Y}_{\nu'}) \\
 \downarrow E'_T & & \downarrow E_T \\
 \text{Hom}(\mathcal{W}_{\nu}, \mathcal{W}_{\nu'}) & \xrightarrow{\Upsilon} & \text{Hom}(\mathcal{W}_{\nu}, \mathcal{W}_{\nu'}).
 \end{array} \tag{6.15}$$

The morphism Υ from Corollary 5.9 comes from the functor $\text{res}: \mathcal{A}(T, N) \text{-mod} \rightarrow \mathcal{A}(T, N_0^{\xi}) \text{-mod}$ and sends the morphism $r_{\nu-\nu'}$ to $r_{\xi}^{-k} r_{\nu-\nu'+k\xi}$ for $k \gg 0$, which is well-defined by the ξ -negativity, and the map (6.11) sends $\mathbb{w}(\lambda, \lambda')$ to $\mathbb{w}(\lambda, \lambda')$. In the left-hand square of (6.18), going right and then down sends

$$\mathbb{w}(\nu, \nu') \mapsto \frac{1}{\Phi_0^{L, N}(\nu, \nu')} r_{\nu-\nu'},$$

while by Lemma 6.11, going down and then to the right sends

$$\mathbb{w}(\nu, \nu') \mapsto \frac{1}{\Phi_0^{L, N_0^{\xi}}(\nu, \nu')} \prod_{\langle \mu, \xi \rangle < 0} \frac{\prod_{\substack{j=0, \dots, \langle \mu, \nu-\nu' \rangle - 1 \\ j \neq \langle \mu, \nu' \rangle}} (\mu + j)}{\prod_{\substack{j=1, \dots, -\langle \mu, \nu-\nu' \rangle \\ j \neq \langle \mu, \nu' \rangle}} (\mu - j)} r_{\xi}^{-k} r_{\nu-\nu'+k\xi}.$$

Now, we must apply (2.5):

$$r_{\xi}^{-k} r_{\nu-\nu'+k\xi} = \prod_{\langle \mu, k\xi \rangle > 0} \prod_{\langle \mu, \nu-\nu' \rangle} \frac{d(\langle \mu, k\xi \rangle, \langle \mu, \nu-\nu' \rangle)}{\prod_{j=1}^{d(\langle \mu, k\xi \rangle, \langle \mu, \nu-\nu' \rangle)} \frac{1}{\mu - j}} \prod_{\langle \mu, k\xi \rangle < 0} \prod_{\langle \mu, \nu-\nu' \rangle} \frac{d(\langle \mu, k\xi \rangle, \langle \mu, \nu-\nu' \rangle) - 1}{\prod_{j=0}^{d(\langle \mu, k\xi \rangle, \langle \mu, \nu-\nu' \rangle) - 1}} \frac{1}{\mu + j} r_{\nu-\nu'}.$$

Note that for $k \gg 0$, this is the same as

$$r_{\xi}^{-k} r_{\nu-\nu'+k\xi} = \prod_{\langle \mu, \xi \rangle > 0} \prod_{\langle \mu, \nu-\nu' \rangle} \frac{-\langle \mu, \nu-\nu' \rangle}{\prod_{j=1}^{-\langle \mu, \nu-\nu' \rangle}} \frac{1}{\mu - j} \prod_{\langle \mu, \xi \rangle < 0} \prod_{\langle \mu, \nu-\nu' \rangle} \frac{\langle \mu, \nu-\nu' \rangle - 1}{\prod_{j=0}^{\langle \mu, \nu-\nu' \rangle - 1}} \frac{1}{\mu + j} r_{\nu-\nu'}, \tag{6.16}$$

which is, as promised, independent of k . On the other hand, we have that

$$\frac{\Phi_0^{L, N_0^\xi}(v, v')}{\Phi_0^{L, N}(v, v')} = \prod_{\langle \mu, \xi \rangle \neq 0} \prod_{j=1, \dots, -\langle \mu, \lambda - \lambda' \rangle} \prod_{j \neq \langle \mu, \lambda' \rangle} (\mu - j). \tag{6.17}$$

Combining equations (6.16) and (6.17), we have the commutativity of this square. □

Lemma 6.14. *Under the hypotheses of 6.12, we have a commutative diagram*

$$\begin{array}{ccc} \widehat{H}^L(\lambda \mathbb{X}_{\lambda'}^L) & \xrightarrow{(6.11)} & \widehat{H}^G(\gamma \mathbb{X}_{\gamma'}) \\ \downarrow E_L & & \downarrow E_G \\ \text{Hom}({}^B\mathcal{W}_{\lambda}^L, {}^B\mathcal{W}_{\lambda'}^L) & \xrightarrow{\Upsilon'} & \text{Hom}({}^B\mathcal{W}_{\gamma}, {}^B\mathcal{W}_{\gamma'}). \end{array}$$

Proof. As in the previous proof, we can equivalently show the commutativity of the diagram:

$$\begin{array}{ccc} \widehat{H}^L(\lambda \mathbb{X}_{\lambda'}^L) & \xrightarrow{(6.11)} & \widehat{H}^G(\gamma \mathbb{X}_{\gamma'}) \\ \downarrow E'_L & & \downarrow E_G \\ \text{Hom}({}^B\mathcal{W}_{\lambda}^L, {}^B\mathcal{W}_{\lambda'}^L) & \xrightarrow{\Upsilon} & \text{Hom}({}^B\mathcal{W}_{\gamma}, {}^B\mathcal{W}_{\gamma'}). \end{array}$$

The morphism Υ from Corollary 5.9 comes from the functor res , which involves passing between modules for the algebras $\mathcal{A}_\varphi^B(G, N)$, $\mathcal{A}_\varphi^{B_L}(L, N)$ and $\mathcal{A}_\varphi^{B_L}(L, N_0^\xi)$.

As we did earlier, let us write $B_L = B_L$ and ${}^{B_L}\mathcal{W}_{\lambda}^{L, N}$ for the weight functor on the category of $\mathcal{A}_\varphi^{B_L}(L, N)$ -modules.

In order to prove the commutativity of the diagram, we break it into two squares

$$\begin{array}{ccccc} \widehat{H}^L(\lambda \mathbb{X}_{\lambda'}^{L, N_0^\xi}) & \xrightarrow{\beta} & \widehat{H}^L(\lambda \mathbb{X}_{\lambda'}^{L, N}) & \xrightarrow{\quad} & \widehat{H}^G(\gamma \mathbb{X}_{\gamma'}) \\ \downarrow E'_L & & \downarrow E_L & & \downarrow E_G \\ \text{Hom}({}^{B_L}\mathcal{W}_{\lambda}^L, {}^{B_L}\mathcal{W}_{\lambda'}^L) & \xrightarrow{\alpha} & \text{Hom}({}^{B_L}\mathcal{W}_{\lambda}^{L, N}, {}^{B_L}\mathcal{W}_{\lambda'}^{L, N}) & \xrightarrow{\quad} & \text{Hom}({}^B\mathcal{W}_{\gamma}, {}^B\mathcal{W}_{\gamma'}). \end{array} \tag{6.18}$$

The map α at the bottom of the left square is induced by the fact that ${}^{B_L}\mathcal{W}_{\lambda}^{L, N}(M) = {}^{B_L}\mathcal{W}_{\lambda}^L(M[r_\xi^{-1}])$, so a natural transformation $f: {}^{B_L}\mathcal{W}_{\lambda} \rightarrow {}^{B_L}\mathcal{W}_{\lambda'}$ applied to $M[r_\xi^{-1}]$ induces a map

$$\alpha(f): {}^{B_L}\mathcal{W}_{\lambda}^{L, N}(M) \rightarrow {}^{B_L}\mathcal{W}_{\lambda'}^{L, N}(M).$$

The top isomorphism β is induced by the isomorphism $\widehat{H}^L(\lambda \mathbb{X}_{\lambda'}^{L, N_0^\xi}) \cong \widehat{H}^L(\lambda \mathbb{X}_{\lambda'}^{L, N})$, which comes from Lemma 6.10. Note that all the maps in this left-hand square are isomorphisms.

We will reduce proving the commutativity of the left square to the Abelian case. The ring \mathbb{A}_T acts freely on the morphism spaces in both categories, so we can check the commutativity after tensor product with \mathbb{L} without loss of generality. Since ξ is W_L -invariant by definition, all the maps in the diagram (6.15) will be W_L -invariant. Thus, by Lemma 6.14, the back square of the cube below commutes:

$$\begin{array}{ccc}
 \bigoplus_{w \in W_L} \widehat{\mathbb{L}} \otimes_{\widehat{\mathcal{A}}_T} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(T, N_0^\xi)}(wv, v') \cdot w & \longrightarrow & \bigoplus_{w \in W_L} \widehat{\mathbb{L}} \otimes_{\widehat{\mathcal{A}}_T} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(T, N)}(wv, v') \cdot w \\
 \nearrow & & \nearrow \\
 \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(L, N_0^\xi)}(v, v') & \longrightarrow & \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(L, N)}(v, v') \\
 \downarrow & & \downarrow \\
 \bigoplus_{w \in W_L} \widehat{\mathbb{L}} \otimes_{\widehat{\mathcal{A}}_T} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(T, N_0^\xi)}(wv, v') \cdot w & \longrightarrow & \bigoplus_{w \in W_L} \widehat{\mathbb{L}} \otimes_{\widehat{\mathcal{A}}_T} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(T, N)}(wv, v') \cdot w \\
 \nearrow & & \nearrow \\
 \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(L, N_0^\xi)}(v, v') & \longrightarrow & \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(L, N)}(v, v')
 \end{array}$$

The horizontal maps along the sides are injective by (6.6) and (6.8), and the sides commute by the compatibility of these isomorphisms. Thus, the front commutes as well, establishing the left-hand square of (6.18).

Now let's concentrate on the right-hand square of (6.18). We will establish this commutativity for all λ, λ' , not necessarily antidominant. To this end, we define a functor $\widehat{\mathcal{X}}^B(L, N) \rightarrow \widehat{\mathcal{X}}^B(G, N)$ which is the identity on objects and on morphisms is given by the natural map

$$\widehat{H}^L(\lambda \mathbb{X}_{\lambda'}^L) \rightarrow \widehat{H}^G(\lambda \mathbb{X}_{\lambda'}^G).$$

When we compare this with the isomorphism (6.8), we find that it is induced by the usual map $W_L \rightarrow W$, since fixed point classes pushforward to fixed point classes.

Next, recall the inclusion of algebras $\mathcal{A}_\varphi^{BL}(L, N) \rightarrow \mathcal{A}_\varphi^B(G, N)$. This map is compatible with the natural map $\widehat{\mathbb{L}} \rtimes \widehat{W}_L \rightarrow \widehat{\mathbb{L}} \rtimes \widehat{W}$ and leads to a morphism

$$\text{Hom}({}^{BL}\mathcal{W}_\lambda^{L, N}, {}^{BL}\mathcal{W}_{\lambda'}^{L, N}) \rightarrow \text{Hom}({}^B\mathcal{W}_\lambda^{G, N}, {}^B\mathcal{W}_{\lambda'}^{G, N}) \tag{6.19}$$

as in Corollary 5.9, based on the description (6.5). Thus, we obtain a functor $\widehat{\mathcal{X}}_{\mathbb{Z}}^{BL}(L, N) \rightarrow \widehat{\mathcal{X}}_{\mathbb{Z}}^B(G, N)$ which is the identity of objects and given on morphisms by (6.19). When $G = L = T$, this map is the identity, and after applying the isomorphism (6.6), is induced the inclusion $W_L \hookrightarrow W$, and the identity map on $\text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(T, N)}(wv, v')$ for each $w \in W_L$. That is, the top and bottom squares of the cube below are commutative:

$$\begin{array}{ccc}
 \bigoplus_{w \in W_L} \widehat{\mathbb{L}} \otimes_{\widehat{\mathcal{A}}_T} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(T, N)}(wv, v') \cdot w & \longrightarrow & \bigoplus_{w \in W} \widehat{\mathbb{L}} \otimes_{\widehat{\mathcal{A}}_T} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(T, N)}(wv, v') \cdot w \\
 \nearrow & & \nearrow \\
 \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(L, N)}(v, v') & \longrightarrow & \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(G, N)}(v, v') \\
 \downarrow & & \downarrow \\
 \bigoplus_{w \in W_L} \widehat{\mathbb{L}} \otimes_{\widehat{\mathcal{A}}_T} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(T, N)}(wv, v') \cdot w & \longrightarrow & \bigoplus_{w \in W} \widehat{\mathbb{L}} \otimes_{\widehat{\mathcal{A}}_T} \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(T, N)}(wv, v') \cdot w \\
 \nearrow & & \nearrow \\
 \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(L, N)}(v, v') & \longrightarrow & \text{Hom}_{\widehat{\mathcal{X}}_{\mathbb{Z}}(G, N)}(v, v')
 \end{array}$$

The commutation of the back square is immediate. This implies that the front square is commutative as well. This is exactly the right-hand square of (6.18), completing the proof. \square

Proof of Theorem 6.12. Now, we wish to reduce from the diagram (6.18) to the corresponding diagram for the spherical algebras:

$$\begin{array}{ccccc}
 \widehat{H}^L(\nu \mathbb{Y}_{\nu'}^{L, N_0^\xi}) & \longrightarrow & \widehat{H}^L(\nu \mathbb{Y}_{\nu'}^{L, N}) & \longrightarrow & \widehat{H}^G(\gamma \mathbb{Y}_{\gamma'}) \\
 \downarrow E'_L & & \downarrow E_L & & \downarrow E_G \\
 \text{Hom}(\mathcal{W}_\nu^L, \mathcal{W}_{\nu'}^L) & \longrightarrow & \text{Hom}(\mathcal{W}_\nu^{L, N}, \mathcal{W}_{\nu'}^{L, N}) & \longrightarrow & \text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'}).
 \end{array} \tag{6.20}$$

First, note that the top line of (6.18) sends $e'(\nu)$ to $e'(\nu)$, so the top line of the right-hand square in (6.20) is just obtained from the top line of (6.18) by multiplying on the left and right by idempotents as in (6.9). On the other hand, the same is true of the bottom row, using (6.10). □

7. Flavoured KLRW algebras

In this section, we study two variants of KLR algebras which we will later connect to Coulomb branch algebras associated to quiver gauge theories. We briefly discuss weighted KLRW algebras (there is already a significant literature on them, for example [Web19b, Web17b, Web19a, BCS17, Bow]). It will be more convenient for us to work with a slight variation on these algebras, which we introduce here, and which may prove to be of independent interest. We call these **flavoured KLRW algebras**, since they allow us to more easily incorporate the flavour parameters of the Coulomb branch, and make a statement more uniform in these parameters. These are close analogues of the metric KLRW algebras which we introduced in [KTW⁺19b] but apply to more general quivers. In Section 8.1, we'll introduce a more general notion of ℓ -flavoured KLRW algebras, which include both the weighted and flavoured algebras defined in this section as special cases.

7.1. Reminder on weighted KLRW algebras

In this section, we remind the reader of the definition of (reduced) weighted KLRW algebras.

Let $\Gamma = (I, E)$ be a quiver, and let $\mathbf{w} \in \mathbb{Z}_{\geq 0}^I$ be a dimension vector. Recall that for an edge $e = i \rightarrow j$, we set $t(e) = i$ and $h(e) = j$. Recall the Crawley-Boevey quiver $\Gamma^{\mathbf{w}}$, defined in Section 3.1. Its vertex set is $I \sqcup \{\infty\}$ and its edge set is the union of the ‘old’ edges $E(\Gamma)$ and w_i ‘new’ edges oriented from i to ∞ . Assume we have chosen a **weighting** of this graph, that is, a map $\vartheta : E(\Gamma^{\mathbf{w}}) \rightarrow \mathbb{R}, e \mapsto \vartheta_e$.

Definition 7.1. A **loading** is a function $\ell : \mathbb{R} \rightarrow I \cup \{0\}$ which is nonzero at finitely many points. Equivalently, a loading is a finite subset of the real line and a labelling of its elements with vertices of Γ .

We call a point $a \in \mathbb{R}$ **corporeal** for the loading ℓ if $\ell(a) \neq 0$, and call $\ell(a)$ its label. We call a point $a \in \mathbb{R}$ **ghostly** for this loading if $\ell(a - \vartheta_e) = h(e)$ for some old edge e , and call e its label. We call the points ϑ_e for the new edges e **red**; these do not depend on the loading.

For our purposes, it will be more convenient to organize this information differently. Let $l_1 < l_2 < \dots < l_n$ be the full list of corporeal values $l_i \in \mathbb{R}$ (i.e. points where $\ell(l_i) \neq 0$). Let $i_k = \ell(l_k)$, giving a sequence $\mathbf{i} = (i_1, \dots, i_n)$. This identification of the corporeal points with the integers $\mathbb{C} = [1, n]$ identifies each ghostly and red point as image of an element of the sets \mathbb{G} and \mathbb{R} , where

$$\mathbb{G} = \{(k, e) \in \mathbb{C} \times E \mid i_k = h(e)\} \quad \mathbb{R} = \{(\star, e) \mid e \in E(\Gamma^{\mathbf{w}}), \infty = h(e)\} \tag{7.1}$$

under the map

$$\alpha_\ell(x) = \begin{cases} l_k & x = k \in \mathbb{C} \\ l_k + \vartheta_e & x = (k, e) \in \mathbb{G} \\ \vartheta_e & x = (\star, e) \in \mathbb{R}. \end{cases}$$

We can think of this map $\alpha_\ell: \text{CGR} \rightarrow \mathbb{R}$ as defined on the union $\text{CGR} = \text{C} \cup \text{G} \cup \text{R}$. We call a loading or loaded sequence **generic** if all the values of α_ℓ are distinct. Throughout the rest of the paper, we will only consider generic loadings. For a generic loading, we can think of G and R as identified with the set of ghostly and red points of the loading. Since we think of red points as ghosts attached to the Crawley-Boevey vertex, we will sometimes want to consider the red and ghostly points together in the set $\text{GR} = \text{G} \cup \text{R}$.

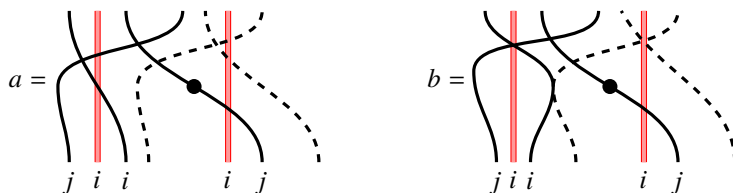
For a generic loading, the map α_ℓ induces an order on CGR , which carries all the important information of the loading. In particular, for different values of ϑ_e , different orders are possible. We'll call the resulting triple $((i_1, \dots, i_n), (l_1, \dots, l_n), <)$ a **loaded sequence**.

Definition 7.2. A **weighted KLRW diagram** is a collection of finitely many oriented smooth curves in $\mathbb{R} \times [0, 1]$; we call these curves **strands**. Each strand must have one endpoint on $y = 0$ and one on $y = 1$, at distinct points from the other strands. These diagrams satisfy the following:

- There is a red strand for each new edge e , with endpoints at $x = \vartheta_e$, and labelled by $t(e) \in I$.
- There are black strands which are not constrained to be vertical but whose projection to the y -axis must be a diffeomorphism onto $[0, 1]$. These are also labelled with vertices in I , and are allowed to carry a finite number of dots.
- For every edge $e \in E(\Gamma)$ with $h(e) = i$, we add a ‘ghost’ of each strand labelled i shifted ϑ_e units to the right (or left if ϑ_e is negative). The ghost is labelled by e and is depicted with a black dotted line. When we need to contrast the original strands with ghosts, we refer to the original strands as ‘corporeal.’

We require that there are no tangencies or triple intersection points between any combination of strands (corporeal, red or ghostly), and no dots on intersection points. We consider these diagrams up to isotopy (relative to the top and bottom) which preserves all these conditions.

For example, if we have an edge $i \rightarrow j$, then the diagram a is a weighted KLRW diagram, whereas b is not since it has a tangency between a strand and a ghost, and two triple points:



Definition 7.3. The degree of a weighted KLRW diagram is the sum of:

1. 2 times the number of dots,
2. -2 times the number of crossings of corporeal strands with the same label and
3. 1 for each crossing of a corporeal strand with label i and a red or ghostly strand with label e , such that $t(e) = i$.

Reading the positions of the corporeal strands along the lines $y = 0$ and $y = 1$, we obtain loadings, which we call the **bottom** and **top** of the diagram. There is a notion of **composition** ab of weighted KLRW diagrams a and b : this is given by stacking a on top of b and attempting to join the bottom of a and top of b . If the loadings from the bottom of a and top of b don't match, then the composition is not defined and by convention is 0, which is not a weighted KLRW diagram, just a formal symbol. This composition rule makes the formal span of all weighted KLRW diagrams over \mathbb{C} into a graded algebra \tilde{T}^ϑ . For each loading ℓ , we have a straight line diagram $e_\ell \in \tilde{T}^\vartheta$, where every horizontal slice is ℓ , and there are no dots.

We will need the notion of equivalent loadings, defined in [Web19b, Definition 2.9]. Informally, two loadings are equivalent when they are isotopic without passing any strand through a relevant ghostly or red point.

Definition 7.4. Let ℓ, ℓ' be two loadings with associated sequences of nodes \mathbf{i}, \mathbf{i}' . We say ℓ, ℓ' are **equivalent** if there is a bijection $\sigma: \mathbb{C} \rightarrow \mathbb{C}$, such that for all $r \in \mathbb{C}$, we have that:

1. $i'_{\sigma(r)} = i_r$
2. for every $(x, e) \in \text{GR}$, such that $i_r = t(e)$, we have that $r < (x, e)$ if and only if $\sigma(r) <' (\sigma(x), e)$ (by convention $\sigma(\star) = \star$).

Definition 7.5. The **weighted KLRW algebra** \tilde{T}^ϑ is the quotient of \tilde{T}^ϑ by relations similar to the original KLR relations but with interactions between differently labelled strands turned into relations between strands and ghosts.

We give the list of local relations below (note that $\vartheta_e < 0$ in all these pictures). Some care must be used when understanding what it means to apply these relations locally. In each case, the LHS and RHS have a dominant term which is related to each other via an isotopy through a disallowed diagram with a tangency, triple point or a dot on a crossing. You can only apply the relations if this isotopy avoids tangencies, triple points and dots on crossings everywhere else in the diagram; one can always choose isotopy representatives sufficiently generic for this to hold.

Another important subtlety is that the relations below are only correct for ϑ generic. If, for example, $\vartheta_e = 0$ for some edge, causing a corporeal strand and its ghost to coincide, then these relations change (but the nongeneric relations can be found by applying a small perturbation to ϑ , and applying the relations below). For example, the usual bigon relation in a KLR algebra (i.e. [KTW⁺19b, (3.1c)]) is a limit of (7.2c–7.2g) below, as $\vartheta_e \rightarrow 0$ for all edges.

$$\begin{array}{c} \bullet \\ \diagdown \\ i \end{array} \begin{array}{c} \diagup \\ j \\ \bullet \end{array} = \begin{array}{c} \diagup \\ i \\ \bullet \end{array} \begin{array}{c} \diagdown \\ j \end{array} \quad \text{for } i \neq j \tag{7.2a}$$

$$\begin{array}{c} \bullet \\ \diagdown \\ i \end{array} \begin{array}{c} \diagup \\ i \end{array} = \begin{array}{c} \diagdown \\ i \end{array} \begin{array}{c} \diagup \\ i \\ \bullet \end{array} + \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ i \end{array} \quad \begin{array}{c} \bullet \\ \diagdown \\ i \end{array} \begin{array}{c} \diagup \\ i \end{array} = \begin{array}{c} \diagup \\ i \\ \bullet \end{array} \begin{array}{c} \diagdown \\ i \end{array} + \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ i \end{array} \tag{7.2b}$$

$$\begin{array}{c} \diagup \\ i \end{array} \begin{array}{c} \diagdown \\ i \end{array} = 0 \quad \text{and} \quad \begin{array}{c} \diagup \\ i \end{array} \begin{array}{c} \diagdown \\ j \end{array} = \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ j \end{array} \tag{7.2c}$$

In all the diagrams below, we assume that i and j are vertices with an edge $e: i \rightarrow j$, and the ghost shown is that attached to e for the strand with label j shown:

$$\begin{array}{c} \text{---} \\ \diagdown \\ k \end{array} \begin{array}{c} \diagup \\ j \end{array} = \begin{array}{c} | \\ k \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ j \end{array} \tag{7.2d}$$

$$\begin{array}{c} \text{---} \\ \diagup \\ k \end{array} \begin{array}{c} \diagdown \\ j \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ k \end{array} \begin{array}{c} | \\ j \end{array} \tag{7.2e}$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (7.2f)$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (7.2g)$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} - \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (7.2h)$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (7.2i)$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (7.2j)$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \delta_{i,j,m} \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (7.2k)$$

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} + \begin{array}{c} \bullet \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (7.2l)$$

For the relations (7.2j) and (7.2l), we also include their mirror images. We also include isotopy through all triple points not shown as relations.

Given $\mathbf{v} \in \mathbb{Z}_{\geq 0}^I$, we let $\tilde{T}_{\mathbf{v}}^{\theta}$ be the subalgebra containing v_i black strands labelled i , for $i \in I$. Since the relations (7.2a–7.21) are homogeneous with respect to the grading induced by the degree of KLRW diagrams given in Definition 7.3, this makes $\tilde{T}_{\mathbf{v}}^{\theta}$ into a graded algebra.

We’ll also be interested in the so-called *steadied* quotients of these algebras. A loading is **unsteady** if there is a group of black strands we can move right arbitrarily far without changing the equivalence class of the loading.

The **steadied quotient** $T_{\mathbf{v}}^{\theta}$ is the quotient of $\tilde{T}_{\mathbf{v}}^{\theta}$ by the two-sided ideal generated by e_{ℓ} , as ℓ ranges over all the unsteady loadings ([Web19b, Definition 2.22]).

Remark 7.6. A reader comparing with the definition of unsteady in [Web19b] might have trouble seeing why this is equivalent. In this paper, we are only interested in the special case discussed in [Web19b, Section 3.1]; that is, we are using the charge c which assigns $c(i) = 1 + i$ for all old vertices and $i - \sum v_i$ to ∞ . The equivalence in this case exactly follows the proof of [Web19b, Theorem 3.6].

Note the contrast in notation here to that introduced below Remark 3.2 in [KTW⁺19b], which used $_{-T}$ for the analogous quotient; since we only use the choice of charge discussed above, we don’t need to contrast with $_{+T}$ which is the steadied quotient for a different charge.

7.2. Flavoured KLRW algebras

We now turn to introducing flavoured KLRW algebras, which can be thought of as variations on the weighted KLRW algebras, that can more aptly describe the representation theory of Coulomb branch algebras $\mathcal{A}(\mathbf{v}, \mathbf{w})$. In this and the following sections, we will also explain the parallels and differences between flavoured KLRW algebras and weighted KLRW algebras. In particular, this will allow us to appeal to results for weighted KLRW algebras which have been developed in [Web19b].

Definition 7.7. A **flavour** on the quiver $\Gamma^{\mathbf{w}}$ is a map $\varphi: E(\Gamma^{\mathbf{w}}) \rightarrow \mathbb{C}, e \mapsto \varphi_e$. A φ -**flavoured sequence** is a triple $(\mathbf{i}, \mathbf{a}, <)$ consisting of:

- o An ordered n -tuple $\mathbf{i} = (i_1, \dots, i_n) \in I^n$.
- o An ordered n -tuple $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{C}^n$ of complex numbers; we call these the **longitudes** of the corresponding vertices in \mathbf{i} .
- o A total order $<$ on CGR (as defined just below (7.1)) extending the usual comparison order on $[1, n]$.

As before, we refer to elements of \mathbb{C} as **corporeal**, elements of \mathbb{R} as **red** and elements of \mathbb{G} as **ghostly**.

A red/ghostly element $g = (x, e) \in \text{GR}$ is endowed with the longitude $a_g = a_x + \varphi_e$ if $x \in \mathbb{C}$, and a red element is endowed with the longitude $a_g = \varphi_e$ if $x = \star$. We require the following properties to be satisfied:

- i. The **real longitudes** $\mathfrak{R}(a_g)$, for $g \in \text{CGR}$, are weakly increasing with respect to $<$.
- ii. If $g \in \text{GR}, m \in \mathbb{C}$ and $\mathfrak{R}(a_g) = \mathfrak{R}(a_m)$, then $g < m$.

Remark 7.8. Note that we had previously used ‘flavour’ to refer to an element of the Lie algebra $\mathfrak{f} = \text{Lie}(F)$, which gives a choice of quantization parameters for $\mathcal{A}(\mathbf{v}, \mathbf{w})$; as discussed in Section 3.1, in the case of a quiver gauge theory corresponding to the data $\Gamma, \mathbf{v}, \mathbf{w}$, we have $F = (\mathbb{C}^{\times})^{E(\Gamma^{\mathbf{w}})}$, so a flavour in the sense defined above is indeed an element of \mathfrak{f} .

For the remainder of the section, fix a flavour φ . A pair of flavoured sequences $(\mathbf{i}, \mathbf{a}, <), (\mathbf{i}', \mathbf{a}', <')$ corresponding to φ are **equivalent** if there is a permutation $\sigma \in S_n$, such that:

1. For all $m \in \mathbb{C}$, we have $i_m = i'_{\sigma(m)}$.
2. For any corporeal $m \in \mathbb{C}$, and $(k, e) \in \text{GR}$, such that $t(e) = i_m$, we have that $m < (k, e)$ if and only if $\sigma(m) <' (\sigma(k), e)$.
3. For any corporeals $k, m \in \mathbb{C}$ with $i_k = i_m$, we have that $\mathfrak{R}(a_k) < \mathfrak{R}(a_m)$ if and only if $\mathfrak{R}(a'_{\sigma(k)}) < \mathfrak{R}(a'_{\sigma(m)})$.

We'll sometimes want to think about the elements of a flavoured sequence one vertex at a time. Fixing a flavoured sequence $(\mathbf{i}, \mathbf{a}, \langle \rangle)$, we'll refer to the set $\{k \in \mathbb{C} \mid i_k = i\}$ as the **corporeals with label** i . Suppose there are v_i corporeals with label i ; this gives a dimension vector $\mathbf{v} \in \mathbb{Z}_{\geq 0}^I$. The longitudes \mathbf{a} can then be organized into a point of $\mathfrak{t} := \prod_{i \in I} \mathbb{C}^{v_i}$.

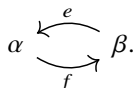
On \mathfrak{t} , we have an action of the Weyl group $W = \prod S_{v_i}$. A point $\gamma \in \mathfrak{t}/W$ will be regarded as a tuple of multisets γ_i of complex numbers, with γ_i of size v_i . Conversely, we can produce flavoured sequences from points of \mathfrak{t}/W .

Lemma 7.9. *Let $\gamma \in \mathfrak{t}/W$. There is a φ -flavoured sequence, unique up to equivalence, for which γ_i is the multiset of longitudes a_k , such that $i_k = i$.*

Proof. We have fixed the multiset of corporeal longitudes (a_1, \dots, a_n) for each vertex i , and the equation $a_g = a_k + \varphi_e$ fixes the longitudes on ghosts associated to the edge e . Choose an order on the union of the multisets γ_i so that the real parts are weakly increasing, and denote the resulting list \mathbf{a} . Define \mathbf{i} so that the k th element in order comes from the set γ_{i_k} .

We now have the associated set CGR, and we let \prec be any order on this set so that the real longitudes are weakly increasing compatible with Property (ii); that is, amongst elements of a fixed real longitude, we put the ghostly/red elements first and then corporeals, using any order within each group. Any two such orders are related by a permutation that only permutes pairs of corporeals with the same real longitude, or elements of GR with the same real longitude. This is manifestly an equivalence. \square

Example 7.10. Let Γ be the Kronecker quiver with cyclic orientation and $\mathbf{w} = 0$. We label vertices by 0 and 1, and edges by e and f :



Suppose we choose constant flavour $\varphi_e = \varphi_f = 1$. Let's consider possible flavoured sequences corresponding to $\mathbf{i} = (\alpha, \beta)$ or $\mathbf{i} = (\beta, \alpha)$. Note that for either choice of \mathbf{i} , there is a unique ghostly element corresponding to each edge, so we may label the ghostlies by e and f . Let a be the longitude associated to vertex α (respectively, b the longitude for β). Then the longitude of the ghost labelled e is $a + 1$ (respectively, of f is $b + 1$). Thus, we can have the following patterns of sequence and longitude:

Sequence	Longitudes	Inequalities
(α, e, β, f)	$(a, a + 1, b, b + 1)$	$\Re(a + 1) \leq \Re(b)$
(α, β, e, f)	$(a, b, a + 1, b + 1)$	$\Re(a) \leq \Re(b) < \Re(a + 1)$
(α, β, f, e)	$(a, b, a + 1, b + 1)$	$\Re(a) = \Re(b)$
(β, α, e, f)	$(a, b, a + 1, b + 1)$	$\Re(a) = \Re(b)$
(β, α, f, e)	$(b, a, b + 1, a + 1)$	$\Re(b) \leq \Re(a) < \Re(b + 1)$
(β, f, α, e)	$(b, b + 1, a, a + 1)$	$\Re(b + 1) \leq \Re(a)$

Any other sequence cannot be flavoured.

Definition 7.11. A **flavoured KLRW diagram** is a collection of finitely many oriented smooth curves (which, as before, we call ‘strands’) whose projection to the y -axis must be a diffeomorphism to $[0, 1]$, labelled with vertices in I . Each strand must have one endpoint on $y = 0$ and one on $y = 1$, at distinct points from the other strands.

The strands are divided into three sets: the **corporeal** (which are drawn as solid black lines), the **red** (which are drawn as solid red lines) and the **ghostly** (which are drawn as dashed black lines). These must satisfy the usual genericity property of avoiding tangencies and triple points between any set of strands.

In addition, a flavoured KLRW diagram carries the data of φ -flavoured sequences $(\mathbf{i}, \mathbf{a}, \langle \rangle)$ and $(\mathbf{i}', \mathbf{a}', \langle \rangle')$ corresponding to the lines $y = 0$ and $y = 1$. These induce bijections between the set of strands

and the sets CGR and CGR' , respectively, by matching the order $<$ with the left-to-right order of strands at $y = 0$, and matching $<'$ with the left-to-right order at $y = 1$. We require that:

1. These bijections are compatible with the division of CGR , CGR' into C , G , R and the set of strands into corporeal, ghostly and red subsets. In particular, the bijection $\text{CGR} \rightarrow \text{CGR}'$ induces a permutation $\sigma \in S_n$ of corporeal strand which respects their labels: $i_m = i'_{\sigma(m)}$.
2. The bijection $\text{GR} \rightarrow \text{GR}'$ between ghostly/red elements in the flavoured sequence is given by $(k, e) \mapsto (\sigma(k), e)$, that is, it is induced by the bijection on corporeals.
3. The longitudes satisfy $a_m - a'_{\sigma(m)} \in \mathbb{Z}$, that is, the difference between the longitudes at the top and bottom of each strand lies in \mathbb{Z} ; if this holds for corporeal strands, then it automatically follows for ghostly/red strands.

We consider these diagrams up to isotopy, which preserves all these conditions. Note that unlike in the weighted case, we allow the points at $y = 0$ and $y = 1$ to move in these isotopies, as long as their order is preserved. Corporeal strands are allowed to carry a finite number of dots.

Let us now describe our conventions for drawing flavoured KLRW diagrams. Strands corresponding to corporeals are drawn as solid black lines, strands corresponding to ghostly elements are drawn as dashed black lines and strands corresponding to red elements as solid red lines. At the top and bottom of each diagram, we include two rows of information:

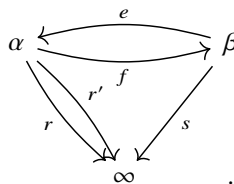
1. In the first row, we write the corresponding vertex i_k for $k \in \text{C}$, the edge e for $(k, e) \in \text{G}$ and the tail vertex $t(e)$ for $(\star, e) \in \text{R}$.
2. In the second row, we write the longitude.

See equation (7.3) for an example of these conventions.

Note that any corporeal (or ghostly) strand does not have a single well-defined longitude but rather two: the longitudes attached to its top and bottom. Red strands, on the other hand, do have a single well-defined longitude. However, the longitudes at the top and bottom of a corporeal or ghostly strand can only differ by an integer, so they give a well-defined element of \mathbb{C}/\mathbb{Z} . In particular, for any pair of strands, the difference between the longitudes at the top of the two strands is integral if and only if the same is true of the longitudes at the bottom of the strands; we say a set of strands where all pairs have this property has **integral difference**.

Example 7.12. Let us return to the example of the Kronecker quiver, with the same conventions as above but the dimension vector

$$v_\alpha = 2 \quad v_\beta = 1 \quad w_\alpha = 2 \quad w_\beta = 1.$$



We fix a flavour φ given by $e, f \mapsto 1$ and $r \mapsto -4, r' \mapsto 0$ and $s \mapsto 2$. We define two φ -flavoured sequences $(\mathbf{i}, \mathbf{a}, <)$ and $(\mathbf{i}', \mathbf{a}', <')$ as follows. First, we set:

$$\mathbf{i} = (\alpha, \alpha, \beta), \mathbf{a} = (-6, -1, 0)$$

$$\mathbf{i}' = (\alpha, \beta, \alpha), \mathbf{a}' = (-3, -2, 3).$$

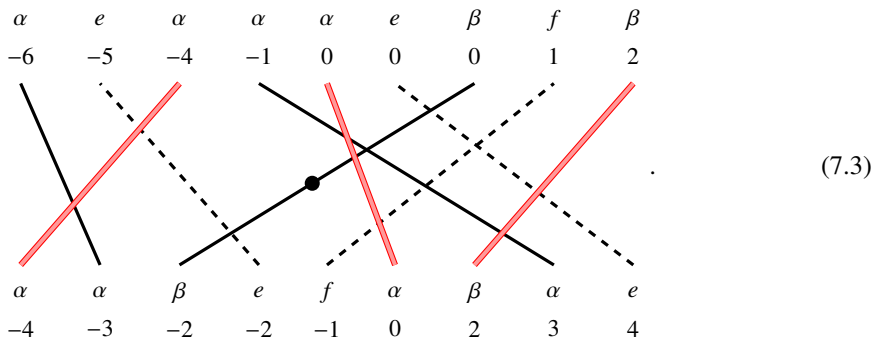
From this, we obtain the corresponding sets of ghostly/red elements:

$$\begin{aligned}
 G &= \{(1, e), (2, e), (3, f)\} & R &= \{(\star, r), (\star, r'), (\star, s)\} \\
 G' &= \{(1, e), (2, f), (3, e)\} & R' &= \{(\star, r), (\star, r'), (\star, s)\}.
 \end{aligned}$$

To ease notation, for $(x, e) \in GR$, we write e_x if $x \in [1, 3]$ and simply e if $x = \star$ (and similarly for elements of G'). We now define $<$ and $<'$ as follows:

$$\begin{aligned}
 1 < e_1 < r < 2 < 3 < r' < e_2 < f_3 < s \\
 r <' 1 <' 2 <' e_1 <' f_2 <' r' <' s <' 3 <' e_3.
 \end{aligned}$$

Here is an example of a flavoured KLRW diagram where the flavour on the top is $(\mathbf{i}, \mathbf{a}, <)$ and the flavour on the bottom is $(\mathbf{i}', \mathbf{a}', <')$:



Focusing, for instance, on the top of the diagram, from the first row, we can read off \mathbf{i} and the total order $<$ on CGR, and from the second row, we can read off the longitudes \mathbf{a} . Note that the longitudes of ghostly/red elements are determined by \mathbf{a} and φ .

Definition 7.13. The **flavoured KLRW algebra** $\tilde{\text{fT}}^\varphi = \tilde{\text{fT}}^\varphi(\Gamma^w)$ is the algebra given by the \mathbb{C} -span of the φ -flavoured KLRW diagrams, modulo isotopy preserving genericity and the local relations (7.2a–7.21) if the set of strands involved in the relation have integral difference. If any pair of strands involved does not have integral difference, then we can simply isotope through a triple point or tangency. As usual, when we multiply diagrams, we must be able to match the flavoured sequences at the top of one diagram and the bottom of the other, or the product is 0 by convention.

Fix a dimension vector $\mathbf{v} \in \mathbb{Z}_{\geq}^I$, which gives the number of corporeal strands with each label, and let $\mathfrak{t} = \prod_i \mathbb{C}^{v_i}$ as above. Let $S \subset \mathfrak{t}$ be any set. We let $\tilde{\text{fT}}_S^\varphi$ be the subalgebra, where we only allow flavoured sequences corresponding to elements of S at the top and bottom of diagrams. We will be particularly interested in the case where $S = \mathcal{O}$ is an orbit of the extended affine Weyl group $\widehat{W} = \prod_i S_{v_i} \times \mathbb{Z}^{v_i}$.

We define a grading on the flavoured KLRW algebra analogous to that on the weighted KLRW algebra: we give a crossing or dot the same grading it would have in the weighted KLRW algebra, except that crossings of strands which don't have integral difference are given degree 0.

Remark 7.14. In [KTW⁺19b], we specialized to a bipartite quiver (with the two sets of vertices called **even** and **odd**), and chose edge orientations to point from even to odd. In that paper, we defined the metric KLRW algebra, which is a special case of the flavoured KLRW algebra defined here.

The most straightforward way to make this connection precise is flavour every old edge in Γ with $1/2$, and each new edge with 0 and assume that every odd/even corporeal has longitude at top or bottom given by $k/2$, where k is an integer of the correct parity. In the conventions of [KTW⁺19b], k would have been the corresponding longitude. Similarly, assume that the longitudes of red strands are of the form $r/2$, where r is an integer of the same parity as the corresponding label. Up to this factor of 2,

we obtain a metric longitude in the sense of [KTW⁺19b, Definition 3.21] from a flavoured sequence in the special case described here, and sets R_i defined by the longitudes on red strands with label i . This defines an isomorphism $\widetilde{\text{fl}}_{\mathcal{S}}^{\varphi} \cong \mathcal{S}^{\mathbf{R}}$, where \mathcal{S} consists of flavoured sequences with the parity convention above, between the flavoured KLRW algebra and the metric KLRW algebra.

This use of half-integers here is in contrast with our usual conventions in this paper. We can apply Lemma 7.21, using the cocycle η with value $1/2$ on even vertices and 0 on odd vertices, to show that the metric KLRW is also equivalent to the flavoured KLRW algebra where all edges are given weight 0 , and all longitudes are integral.

Definition 7.15. Let $e(\mathbf{i}, \mathbf{a}, <)$ denote the idempotent given by the straight-line diagram with the flavoured sequence $(\mathbf{i}, \mathbf{a}, <)$. Given $\gamma \in \mathfrak{t}/W$, let $e(\gamma) = e(\mathbf{i}, \mathbf{a}, <)$ for the flavoured sequence associated to γ by Lemma 7.9.

As in [Web19b, Proposition 2.15], there is a natural symmetry in the definition of flavoured KLRW algebras. We may view a flavour φ as a 1-cocycle on the graph $\Gamma^{\mathbf{w}}$. Let $\eta : I \sqcup \{\infty\} \rightarrow \mathbb{C}$ be a 0-cocycle with $\eta_{\infty} = 0$. Then we may define a cohomologous 1-cocycle $\varphi - d\eta$, by $(\varphi - d\eta)_e = \varphi_e - \eta_{h(e)} + \eta_{t(e)}$. Given an orbit $\mathcal{S} \subset \prod_i \mathbb{C}^{v_i}$, we may define a new orbit $\mathcal{S} + \eta$ by simultaneously translating the \mathbb{C}^{v_i} components by η_i .

Lemma 7.16. *With notation as above, there is an isomorphism*

$$\widetilde{\text{fl}}_{\mathcal{S}}^{\varphi} \cong \widetilde{\text{fl}}_{\mathcal{S}+\eta}^{\varphi-d\eta}$$

defined by shifting longitudes and reordering as necessary.

Proof. As in [Web19b, Proposition 2.15]³ under this isomorphism, a corporeal strand with label i has its longitudes at top and bottom both shifted by η_i , while a ghostly/red strand labelled by an edge e has its longitude shifted by $\eta_{t(e)}$. Since all corporeal strands labelled by $t(e)$ also have their longitudes shifted by $\eta_{t(e)}$, this shifting preserves all crossings between ghostly/red strands labelled e and corporeal strands labelled $t(e)$. □

Definition 7.17. By analogy with loadings, we call a flavoured sequence $(\mathbf{i}, \mathbf{a}, <)$ **unsteady** if, for some $0 < k < n$, the last k elements of CGR consists of a group of corporeal elements and all their ghosts. Importantly, this group should not contain any ghost of one of the corporeal strands which doesn't lie in it, nor any red strands. We define $\widetilde{\text{fl}}_{\mathcal{S}}^{\varphi}$, the **steadied quotient of the algebra** $\widetilde{\text{fl}}_{\mathcal{S}}^{\varphi}$, to be the quotient of $\widetilde{\text{fl}}_{\mathcal{S}}^{\varphi}$ by the two-sided ideal generated by all the idempotents for unsteady flavoured sequences.

Example 7.18. Consider the situation of Example 7.12, and assume $w_{\alpha} = 1, w_{\beta} = 0$, so there is one element of \mathbf{R} , which we denote r . In this case, (β, f, r, α, e) is unsteady since the last two entries are a strand and its only ghost; similarly, (r, β, α, f, e) is unsteady because of its last four entries. On the other hand, (β, r, α, f, e) is not unsteady.

7.3. Reduction to the integral case

Definition 7.19. For a given orbit $\mathcal{S} \subset \mathfrak{t}$, we let $\widetilde{\Gamma}$ be the subgraph of $\Gamma \times (\mathbb{C}/\mathbb{Z})$ consisting of the set \widetilde{I} of pairs $(i, [z])$, for $z \in \mathbb{C}$ which appear as a coordinate in the factor \mathbb{C}^{v_i} for some element of \mathcal{S} . This has an adjacency $(i, [z]) \rightarrow (j, [w])$ for each edge $e = i \rightarrow j$ with $\varphi_e \equiv z - w \pmod{\mathbb{Z}}$.

We have an induced dimension vector $\widetilde{\mathbf{v}} : E(\widetilde{\Gamma}) \rightarrow \mathbb{Z}_{\geq 0}$ defined as follows. Let $x = (x_i) \in \mathfrak{t}$ be any element of \mathcal{S} . Then $\widetilde{v}_{i,[z]}$ is the number of entries in x_i whose class in \mathbb{C}/\mathbb{Z} is equal to $[z]$. Note that we have a canonical isomorphism $\mathbb{C}^{v_i} \cong \prod_{[z]} \mathbb{C}^{\widetilde{v}_{i,[z]}}$, identifying the coordinates (i, k) , such that $x_{i,k} \equiv z \pmod{\mathbb{Z}}$ with the coordinates of $\mathbb{C}^{\widetilde{v}_{i,[z]}}$. Thus, we can naturally consider x as an element of

³Note that the published version of this paper has a sign error, and should read ' $\vartheta - d\eta$ ', not ' $\vartheta + d\eta$ '.

$\prod_{i,[z]} \mathbb{C}^{\tilde{v}_{i,[z]}}$. Let $\tilde{\mathcal{S}}$ be the orbit under the Weyl group of $\tilde{\Gamma}$ of x ; this is the same as the elements of \mathcal{S} , whose projection to $\mathbb{C}^{\tilde{v}_{i,[z]}}$ lies in $(\mathbb{Z} + z)^{\tilde{v}_{i,[z]}}$.

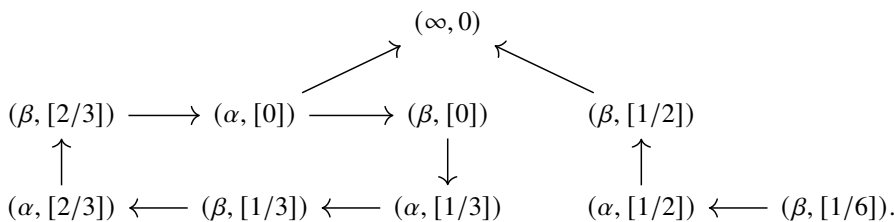
We will also need an induced vector $\tilde{\mathbf{w}}$; this can be read off by adding the Crawley-Boevey vertex $(\infty, 0)$ to $\tilde{\Gamma}$ and applying the same rules as above to new edges. Thus, $\tilde{w}_{i,[z]}$ is the number of new edges with flavour lying in the coset $[z]$. Note that unlike $\tilde{\mathbf{v}}$, the sum of the entries of $\tilde{\mathbf{w}}$ might be less than that for \mathbf{w} , since there might be flavours on new edges not congruent to any coordinate of an element of \mathcal{S} ; in fact, for a generic orbit \mathcal{S} , we will have $\tilde{\mathbf{w}} = 0$.

Finally, we also have an induced flavour $\tilde{\varphi}$ on $\tilde{\Gamma}$ by pullingback φ by the projection map $\tilde{\Gamma} \rightarrow \Gamma$, that is, the flavour of $(i, [z]) \rightarrow (j, [w])$ is equal to φ_e , where $e = i \rightarrow j$.

Example 7.20. Consider the Kronecker quiver with conventions as in Example 7.12, with $v_\alpha = 5$, $v_\beta = 6$ and \mathcal{S} is the orbit containing the elements $(0, 1/3, 1/2, 2/3, 2/3) \in \mathbb{C}^{v_\alpha}$ and $(0, 1/6, 1/3, 1/3, 1/2, 2/3) \in \mathbb{C}^{v_\beta}$. If we choose the flavours

$$\varphi_e = 1/3 \quad \varphi_f = 0 \quad \varphi_r = 0 \quad \varphi_{r'} = \sqrt{2} \quad \varphi_s = 1/2,$$

then every component of $\tilde{\Gamma}$ lies in a union of 6-cycles obtained as 3-fold covers of the Kronecker quiver. In this case, we obtain one full 6-cycle, and three vertices from another, with the Crawley-Boevey graph drawn below. Note that since $(\alpha, \sqrt{2})$ is not a vertex ($\sqrt{2}$ is not a coordinate in the correct orbit), r' does not contribute to $\tilde{\mathbf{w}}$.



The nonzero values of the dimension vectors are:

$$v_{\alpha,[0]} = 1 \quad v_{\alpha,[1/3]} = 1 \quad v_{\alpha,[1/2]} = 1 \quad v_{\alpha,[2/3]} = 2 \quad v_{\beta,[0]} = 1 \quad v_{\beta,[1/6]} = 1$$

$$v_{\beta,[1/3]} = 2 \quad v_{\beta,[1/2]} = 1 \quad v_{\beta,[2/3]} = 1 \quad w_{\alpha,[0]} = 1 \quad w_{\beta,[1/2]} = 1.$$

Lemma 7.21. Let $\mathcal{S} \subset \prod_i \mathbb{C}^{v_i}$ be an orbit. Let $\tilde{\Gamma}, \tilde{\varphi}, \tilde{\mathbf{v}}, \tilde{\mathbf{w}}$ be as above.

1. We have an isomorphism of algebras $\tilde{\text{fl}}_{\mathcal{S}}^{\varphi}(\Gamma^{\mathbf{w}}) \cong \tilde{\text{fl}}_{\mathcal{S}}^{\tilde{\varphi}}(\tilde{\Gamma}^{\tilde{\mathbf{w}}})$.
2. There is an isomorphism of algebras

$$\tilde{\text{fl}}_{\mathcal{S}}^{\varphi}(\Gamma^{\mathbf{w}}) \cong \tilde{\text{fl}}_{\mathcal{S}'}^{\varphi'}(\tilde{\Gamma}^{\tilde{\mathbf{w}}}),$$

where φ' is an integral flavour on $\tilde{\Gamma}$, such that $\varphi' = \tilde{\varphi} - d\eta$ for some 0-cocycle η on $\tilde{\Gamma}$, and where $\mathcal{S}' = \prod_{i,[z]} \mathbb{Z}^{\tilde{v}_{i,[z]}}$.

Proof.

1. If we have a flavoured sequence with longitude in the orbit $\mathcal{S} \subset \prod_i \mathbb{C}^{v_i}$, we can canonically lift this to a flavoured sequence in $\tilde{\Gamma}$, by giving an element of CGR with label i and longitude a the label $(i, [a]) \in \tilde{\Gamma}$, and leaving the longitude unchanged. This defines a homomorphism $\tilde{\text{fl}}_{\mathcal{S}}^{\varphi}(\Gamma) \rightarrow \tilde{\text{fl}}_{\mathcal{S}}^{\tilde{\varphi}}(\tilde{\Gamma})$. On the other hand, if we have a diagram in $\tilde{\text{fl}}_{\mathcal{S}}^{\tilde{\varphi}}(\tilde{\Gamma})$, then by assumption, any strand with label $(i, [a])$ has longitude in $[a]$, so we can define an inverse map just turning the label to i , keeping the longitude the same.

2. A choice of flavour defines a 1-cocycle on the graph Γ^w , which defines a class $\beta \in H^1(\tilde{\Gamma}; \mathbb{C}/\mathbb{Z})$. The equation $\varphi_e \equiv z - w \pmod{\mathbb{Z}}$ exactly guarantees that β is the coboundary of the \mathbb{C}/\mathbb{Z} -valued 0-cocycle sending $(i, [z]) \mapsto -[z]$. In other words, β is trivial; one can understand $\tilde{\Gamma}$ as the unique minimal cover of Γ with this property. Choose a \mathbb{C} -valued 0-cycle η on $\tilde{\Gamma}$ with the property that for each vertex (i, ξ) , we have that $[\eta(i, \xi)] = \xi$. In this case, we have that $\varphi' := \tilde{\varphi} - d\eta$ is integer valued, and we also achieve $\mathcal{S}' = \mathcal{S} + \eta$.

By Part (1) together with Lemma 7.16, we have isomorphisms:

$$\tilde{\text{fl}}_{\mathcal{S}'}^{\varphi}(\Gamma) \cong \tilde{\text{fl}}_{\mathcal{S}}^{\tilde{\varphi}}(\tilde{\Gamma}) \cong \tilde{\text{fl}}_{\mathcal{S}+\eta}^{\tilde{\varphi}-d\eta}(\tilde{\Gamma}) \cong \tilde{\text{fl}}_{\mathcal{S}'}^{\varphi'}(\tilde{\Gamma}). \quad \square$$

7.4. Connecting flavoured KLRW and weighted KLRW algebras

As promised, we will lay out here the parallels between the weighted and flavoured approaches. We have a close analogy based on equating:

Weighted KLRW	Flavoured KLRW
weighting	flavour
loading/loaded sequence	flavoured sequence
weighted KLRW diagram	flavoured KLRW diagram
positions of strands	longitudes

The key differences here are:

1. Loaded sequences are necessarily valued in the real numbers, and exactly match the x -values of the relevant KLRW diagram. In particular, any generic horizontal slice of a weighted KLRW diagram gives a loading, and so we must be able to deform these continuously. Small deformations of a weighted KLRW diagram genuinely change the underlying loadings.
2. Flavoured sequences can be valued in the complex numbers or even in more general sets (see Definition 8.1). A flavoured KLRW diagram only has well-defined longitudes at the top and bottom, and a slice in the middle has no fixed flavoured sequence, and might correspond to an order that is not compatible with any flavoured sequence. Integrality plays an important role in flavoured KLRW algebras, since the relations depend on whether strands have integral difference; there is no corresponding notion for weighted KLRW algebras.

Philosophically, weighted KLRW algebras capture the behaviour of flavoured KLRW algebras in the case where all flavours are on edges and all longitudes are integers. Thus, we assume this integrality for the remainder of this section, and let $\tilde{\text{fl}}_{\mathcal{V}}^{\varphi} = \tilde{\text{fl}}_{\mathcal{S}}^{\varphi}$, where $\mathcal{S} = \prod_i \mathbb{Z}^{v_i} \subset \prod_i \mathbb{C}^{v_i}$. We'll compare with the weighted KLRW algebra $\tilde{T}_{\mathcal{V}}^{\vartheta}$ for the same Dynkin diagram, with weights $\vartheta_e = \varphi_e - 1/2$.

In this case, we make a precise connection between weighted and flavoured KLRW algebras. This begins with a precise correspondence between flavoured and loaded sequences. The underlying idea is simply to think of the flavoured sequence as a loaded sequence, but we need to perturb this definition a small amount. Given flavoured sequence $(\mathbf{i}, \mathbf{a}, <)$ with integral longitudes, we consider the loaded sequence $\ell(\mathbf{i}, \mathbf{a}, <) = (\mathbf{i}, \ell, <')$, where

$$l_k = a_k + k\epsilon \quad \text{for some } 0 < \epsilon \ll \frac{1}{2n}$$

and $<'$ is the order induced by the function $\alpha_{\ell}: \text{CGR} \rightarrow \mathbb{R}$. Note that this order is independent of ϵ given our upper bound on it. The x values of ghostly/red points are given by $\alpha_{\ell}(k, e) = l_k + \vartheta_e = l_k + \varphi_e - 1/2$, which is the longitude of (k, e) minus $\frac{1}{2} - k\epsilon$.

Lemma 7.22. *The flavoured sequence $(\mathbf{i}, \mathbf{a}, <)$ is equivalent to the sequence $(\mathbf{i}, \mathbf{a}, <')$ using the order from the loaded sequence $\ell(\mathbf{i}, \mathbf{a}, <)$.*

Proof. Consider two elements $x, y \in \text{CGR}$, then we need to check that the relative order is the same for the flavoured and loaded sequences whenever:

1. $x = k, y = m \in \mathbb{C}$ with $k < m$: in this case, $a_k \leq a_m$, so

$$l_k = a_k + k\epsilon \leq a_m + k\epsilon < a_m + m\epsilon = l_m.$$

2. $x = k \in \mathbb{C}, y = (m, e) \in \mathbb{G}$: in this case, we have $x > y$ if $a_k \geq a_m + \varphi_e$ and $x < y$ if $a_k < a_m + \varphi_e$. On the other hand, we have

$$\alpha_\ell(k) - \alpha_\ell(m, e) = l_k - l_m - \vartheta_e = a_k - a_m - \varphi_e + (k - m)\epsilon + \frac{1}{2}.$$

Since $a_k - a_m - \varphi_e$ is an integer, for ϵ sufficiently small, this is positive if $a_k \geq a_m + \varphi_e$ and negative otherwise. □

This shows that every flavoured sequence has a loaded sequence which gives the same order. However, the opposite is not true: there can be loaded sequences not equivalent to those coming from any flavoured sequence, due to the integrality requirements. The existence of nonparity idempotents in [KTW⁺19b] is an example of this phenomenon.

Consider diagrams which interpolate between flavoured and weighted diagrams: they should obey all the requirements of a flavoured diagram but only have a choice of flavoured sequence at $y = 1$, whereas at $y = 0$, they satisfy the weighted condition that the distance between strands and ghosts is exactly given by the weights. The result is a $\widetilde{\text{fl}}_\mathbb{V}^\varphi\text{-}\widetilde{\text{T}}_\mathbb{V}^\vartheta$ bimodule $\widetilde{\text{FW}}$ over the weighted and flavoured KLRW algebras. This is the special case of the bimodule relating KLRW algebras flavoured by different sets given in Definition 8.6; in particular, Lemma 8.7 carefully verifies that this bimodule structure is well-defined.

In this bimodule, we can form a straight-line diagram in $\widetilde{\text{FW}}$ joining the loading $\ell(\mathbf{i}, \mathbf{a}, <)$, and the flavoured sequence $(\mathbf{i}, \mathbf{a}, <)$ at the bottom.

Theorem 7.23. *Let $(\mathbf{i}, \mathbf{a}, <)$ be a flavoured sequence, and set $\ell = \ell(\mathbf{i}, \mathbf{a}, <)$. We have an isomorphism $e(\mathbf{i}, \mathbf{a}, <)\widetilde{\text{FW}} \cong e(\ell)\widetilde{\text{T}}_\mathbb{V}^\vartheta$. The functor $\widetilde{\text{FW}} \otimes_{\widetilde{\text{T}}_\mathbb{V}^\vartheta} -$ realizes the category of modules over the flavoured KLRW algebra $\widetilde{\text{fl}}_\mathbb{V}^\varphi$ as a quotient of the category of modules over the weighted KLRW algebra $\widetilde{\text{T}}_\mathbb{V}^\vartheta$ by the subcategory of modules killed by all loadings that correspond to an integral flavoured sequence.*

Proof. Any diagram in $e(\mathbf{i}, \mathbf{a}, <)\widetilde{\text{FW}}$ can be factored into the straight line diagram joining $(\mathbf{i}, \mathbf{a}, <)$ to ℓ at the bottom and an element of the weighted KLRW algebra at the top. This gives the isomorphism $e(\mathbf{i}, \mathbf{a}, <)\widetilde{\text{FW}} \cong e(\ell)\widetilde{\text{T}}_\mathbb{V}^\vartheta$.

This shows that as a right module over the weighted KLRW algebra, $\widetilde{\text{FW}}$ is a projective module with endomorphisms given by the flavoured KLRW algebra $\widetilde{\text{fl}}_\mathbb{V}^\varphi$. The result follows. □

The failure of this functor to be a Morita equivalence is a ‘degenerate’ property, which happens for a relatively small set of flavours φ . These are analogous to aspherical parameters for Cherednik algebras or singular central characters of $U(\mathfrak{gl}_n)$ (this analogy can be made precise by realizing these algebras as Coulomb branches).

Remark 7.24. This result can be extended to the nonintegral case, using the isomorphism of Lemma 7.21 so that $\widetilde{\text{fl}}_\mathcal{S}^\varphi$ is isomorphic to a flavoured KLRW algebra (possibly for a different graph) with integral flavours and longitudes.

We can construct a steadied quotient $\text{FW} = \text{fl}_\mathbb{V}^\varphi \otimes_{\widetilde{\text{fl}}_\mathbb{V}^\varphi} \widetilde{\text{FW}} \otimes_{\widetilde{\text{T}}_\mathbb{V}^\vartheta} T_\mathbb{V}^\vartheta$ of $\widetilde{\text{FW}}$ as well.

Proposition 7.25. *For any flavoured sequence $(\mathbf{i}, \mathbf{a}, <)$ with $\ell = \ell(\mathbf{i}, \mathbf{a}, <)$, we have an isomorphism $e(\mathbf{i}, \mathbf{a}, <)\text{FW} \cong e(\ell)T_\mathbb{V}^\vartheta$. The functor $\text{FW} \otimes -$ realizes the modules over the steadied flavoured KLRW algebra $\text{fl}_\mathbb{V}^\varphi$ as a quotient of modules over the steadied weighted KLRW algebra $T_\mathbb{V}^\vartheta$ by the subcategory of modules killed by all loadings that carry an integral flavouring.*

Proof. Let $I_1 \subset \widetilde{\Pi}_v^\varphi$ and $I_2 \subset \widetilde{T}_v^\theta$ be the kernels of the maps to the steadied quotients. Consider the module $e(\mathbf{i}, \mathbf{a}, \langle \rangle)FW$. This is, by definition, the quotient of $e(\mathbf{i}, \mathbf{a}, \langle \rangle)\widetilde{FW} = e(\ell)\widetilde{T}_v^\theta$ by the submodule $e(\mathbf{i}, \mathbf{a}, \langle \rangle)I_1\widetilde{FW} + e(\mathbf{i}, \mathbf{a}, \langle \rangle)\widetilde{FW}I_2$. Of course, $e(\ell)\widetilde{T}_v^\theta$ is the quotient by $e(\mathbf{i}, \mathbf{a}, \langle \rangle)\widetilde{FW}I_2 \cong e(\ell)\widetilde{T}_v^\theta I_2$. Thus, we only need to prove that $e(\mathbf{i}, \mathbf{a}, \langle \rangle)I_1\widetilde{FW} \subset e(\mathbf{i}, \mathbf{a}, \langle \rangle)\widetilde{FW}I_2$. The submodule $e(\mathbf{i}, \mathbf{a}, \langle \rangle)I_1\widetilde{FW}$ is spanned by diagrams of the form aeb , where $e \in \widetilde{\Pi}_v^\varphi$ is an unsteady idempotent. If ℓ' is the corresponding loading, then by our proof above, we can write $eb = me(\ell')b'$, where m is the straight line diagram joining e to $e(\ell)$ and b' is the image of b under the isomorphism $e\widetilde{FW} \cong e(\ell)\widetilde{T}_v^\theta$. Since ℓ' is also unsteady, $e(\ell')b' \in I_2$ and $aeb = ame(\ell')b' \in e(\mathbf{i}, \mathbf{a}, \langle \rangle)\widetilde{FW}I_2$.

This shows that $e(\mathbf{i}, \mathbf{a}, \langle \rangle)FW \cong e(\ell)\widetilde{T}_v^\theta$. The rest of the proof is identical to that of Theorem 7.23. \square

7.5. Connecting flavoured KLRW algebras and cyclotomic KLR algebras

Choose a large integer $H \gg 0$. For any sequence $\mathbf{i} \in I^n$ with v_i occurrences of i , we have a corresponding flavoured sequence with $(\mathbf{i}, \mathbf{h}, \langle \rangle)$, where $\mathbf{h} = (H, 2H, \dots, nH)$. We let $e(\mathbf{i}, H)$ be the idempotent corresponding to this flavoured sequence (see Definition 7.15). Similarly, $e(\mathbf{i}, -H)$ denotes the idempotent defined as above, except using $-H$.

We'll need to consider the usual KLR algebra [KL11, Rou]; the precise presentation we want is given by [Web17a, Definition 2.5], with $Q_{ij}(u, v) = (u - v)^{\#j \rightarrow i} (v - u)^{\#i \rightarrow j}$. This is another diagrammatic algebra, which we can describe as a special case of the KLRW algebra we defined above (as suggested by the name, this is the opposite of the historical order these were introduced). In this special case:

- we take all weights to be 0, that is, all ghosts coincide with the corresponding corporeal strand.
- we have no red strands, that is, no edges connecting to the Crawley-Boevey vertex.

We let R_v denote this algebra in the case where there are v_i strands of label i . The cyclotomic quotient R_v^w of R_v is the quotient of this algebra by the two-sided ideal generated by w_i dots on the right-most strand. By [Web17a, Theorem 4.18], this is the same as T_v^0 , where we include w_i red strands labelled i , all of whom have weight 0.

For any fixed integral flavour φ , let $e_H = \sum_i e(\mathbf{i}, -H) \in \widetilde{\Pi}_v^\varphi$ be the sum of these idempotents. By Proposition 7.25 and [Web19b, Theorem 3.6], we have that:

Proposition 7.26. *The algebra $e_H \widetilde{\Pi}_v^\varphi e_H$ is isomorphic to the cyclotomic KLR algebra R_v^w for the quiver Γ .*

Thus, as usual, $M \mapsto e_H M$ is a quotient functor, realizing R_v^w -mod as a quotient of $\widetilde{\Pi}_v^\varphi$ -mod by the modules killed by e_H . Note that this result is independent of φ , so this captures a part of the category insensitive to this flavour beyond its integrality.

Let \mathcal{S} be a fixed orbit. The algebra $\widetilde{\Pi}_\mathcal{S}^\varphi(\Gamma^w)$ is nonzero, but often it will have trivial steady quotient. We can precisely describe when this is the case by considering the perspective of categorification. As in Definition 7.19, we use \mathcal{S} to define a new quiver $\widetilde{\Gamma}$. Let λ be a highest weight of $\mathfrak{g}_{\widetilde{\Gamma}}$, such that $\alpha_i^\vee(\lambda) = \tilde{w}_i$. Let $\mu = \lambda - \sum_{j \in \widetilde{\Gamma}} \tilde{v}_j \alpha_j$. One consequence of the categorification theorem for cyclotomic KLR algebras ([Web17a, Theorem 3.21]) is that the cyclotomic KLR algebra $R_{\widetilde{\Gamma}}^w$ is nonzero if and only if the μ -weight space of the simple $\mathfrak{g}_{\widetilde{\Gamma}}$ -module $V(\lambda)$ is nonzero. Thus, reducing the integral case with Lemma 7.21 and applying Proposition 7.26, we find:

Corollary 7.27. *The algebra $\widetilde{\Pi}_\mathcal{S}^\varphi(\Gamma^w)$ is nonzero if the μ -weight space of $V(\lambda)$ is nonzero.*

Remark 7.28. The ‘only if’ direction of this theorem is also true, but it’s a bit outside the scope of this paper to prove. The most straightforward approach is to consider the deformation of the steadied flavoured KLRW algebra analogous to the approach [Webc]. This allows us to reduce to the case of a generic flavour, where the idempotent e_H will induce a Morita equivalence to the cyclotomic KLR algebra.

While we won’t prove the full ‘only if’ direction, we do require one weaker version of it.

Lemma 7.29. *If there is a component of $\tilde{\Gamma}$ that does not contain a vertex $(i, [z])$ with $\tilde{w}_{i,[z]} > 0$, then $\mathfrak{fl}_{\mathcal{S}}^{\varphi}(\Gamma^{\mathbf{w}}) = 0$.*

Proof. As usual, we reduce to the integral case, so we can assume there is a component C of Γ where \mathbf{w} vanishes. For a fixed $\gamma \in (\mathfrak{t} + \varphi)/W$, let γ_H be the resulting element, where we add an integer H to each coordinate corresponding to a vertex in C . Let θ denote the diagram with $e(\gamma)$ at the bottom and $e(\gamma_H)$ at the top, with strands joining terminals that correspond to the same coordinate. For $H \gg 0$, this has the effect of moving all strands with labels in C to the right, and all other strands to the left while introducing a minimal number of crossings. Let θ' be the reflection of this diagram through a horizontal line. For $H \gg 0$, the idempotent $e(\gamma_H)$ is unsteady, so θ and θ' are both zero in the steadied quotient.

The relations (7.2c–7.2d) show that for any H , we have $\theta'\theta = e(\gamma)$ (this is where we use that C has no vertex with $w_i > 0$). Thus, $e(\gamma)$ is zero in $\mathfrak{fl}_{\mathcal{S}}^{\varphi}(\Gamma^{\mathbf{w}})$. Since γ was arbitrary, all idempotents vanish in $\mathfrak{fl}_{\mathcal{S}}^{\varphi}(\Gamma^{\mathbf{w}})$ and thus the algebra must be 0. \square

8. Induction and restriction for flavoured KLRW algebras

8.1. Induction and restriction bimodules

We will now define induction and restriction functors for flavoured KLRW algebras, paralleling those for other versions of KLRW algebras. For this construction, we will need a more general version of flavoured KLRW algebras, where the longitudes are allowed to take values in a more general set rather than \mathbb{C} . Let ℓ be a set equipped with:

1. An action of a partially ordered Abelian group $(A, +, \leq)$. We denote the action of $a \in A$ on $x \in \ell$ by $x + a$.
2. A partial preorder \leq on ℓ compatible with the order on A , that is, satisfying $x + a \leq x + a'$ if $a \leq a'$.
3. An equivalence relation \sim on ℓ , such that \leq defines a total order on each equivalence class.
4. A basepoint $* \in \ell$.

Definition 8.1. An ℓ -flavouring of an oriented graph Γ is an assignment of an element $\varphi_e \in A$ for each edge e of $\Gamma^{\mathbf{w}}$. A ℓ -flavoured sequence for a flavoured graph is a triple $(\mathbf{i}, \mathbf{a}, <)$ consisting of an n -tuple $\mathbf{i} \in I^n$, a choice of longitude $\mathbf{a} \in \ell^n$ and an order $<$ on the set CGR (defined as before, see (7.1)). We define the longitude of $g = (x, e) \in \text{GR}$ to be $a_g = a_x + \varphi_e$ if $x \in \mathbb{C}$, and $a_g = * + \varphi_e$ if $x = \star$. We require the properties:

- i. If $g, g' \in \text{CGR}$ and $g > g'$, then $a_g \not\sim a_{g'}$.
- ii. If $g \in \text{GR}, m \in \mathbb{C}$ and $a_g \approx a_m$ (that is, $a_g \geq a_m$ and $a_m \geq a_g$), then $g < m$.

The cases where we want to apply this are:

ℓ	A	\leq	\sim	$*$
\mathbb{C}	\mathbb{C}	$z \leq z'$ iff $\Re(z) \leq \Re(z')$	$z \sim z'$ iff $z - z' \in \mathbb{Z}$	0
\mathbb{R}	\mathbb{R}	\leq	$a \sim b \forall a, b$	0
$\mathbb{Z} \times \mathbb{C}$	\mathbb{C}	lexicographic	$(m, z) \sim (m', z')$ iff $m = m'$ and $z - z' \in \mathbb{Z}$	$(0, 0)$

By ‘lexicographic,’ we mean that $(m, x) \geq (n, y)$ if $m > n$ or if $m = n$ and $\Re(x) \geq \Re(y)$.

The first of these cases, $\ell = \mathbb{C}$, recovers the definition of flavoured sequences from Definition 7.7. The second case, $\ell = \mathbb{R}$, recovers the definition of loaded sequences. The third case, where $\ell = \mathbb{Z} \times \mathbb{C}$, will be the main case of interest in this section.

We define the ℓ -flavoured KLRW diagrams and algebra exactly as in Definitions 7.11 and 7.13, with the top and bottom of the diagram now having ℓ -flavoured sequences. That is:

Definition 8.2. Fix a flavour φ . A ℓ -flavoured KLRW diagram (corresponding to φ) is a collection of finitely many oriented smooth curves satisfying the conditions of Definition 7.11. The only change is that we must now label the top and bottom with the data of ℓ -flavoured sequences $(\mathbf{i}, \mathbf{a}, <)$ and $(\mathbf{i}', \mathbf{a}', <')$, respectively. There is one important change to the compatibility conditions in Definition 7.11: instead of requiring integral difference between the labels at the top and bottom of a strand, we require them to be equivalent under \sim .

Definition 8.3. The ℓ -flavoured KLRW algebra $\widetilde{\text{fT}}_\ell$ is the algebra given by the \mathbb{C} -span of the ℓ -flavoured KLRW diagrams, modulo isotopy preserving genericity and the local relations (7.2a–7.21) if the strands involved in the relation all have longitudes in the same equivalence class under \sim ; if any pair of strands involved have longitudes not equivalent under \sim , then we can simply isotope through a triple point or tangency. As usual, when we multiply diagrams, we must be able to match the flavoured sequences at the top of one diagram and the bottom of the other, or the product is 0 by convention.

Remark 8.4. One reason that this definition is useful, though we will not develop it in this paper, is to study the representation theory of Coulomb branches over fields \mathbb{k} of characteristic 0 other than \mathbb{C} . The appropriate object describing this on the ‘KLR side’ is a \mathbb{k} -flavoured KLRW algebra for a preorder on \mathbb{k} refining the partial order, where $a \leq b$ if and only if $b - a \in \mathbb{Z}_{\geq 0}$.

As one would expect, the cases $\ell = \mathbb{C}$ and $\ell = \mathbb{R}$ discussed in the table above recover the flavoured and weighted KLRW algebras from Section 7. Since we have already covered these algebras in detail, we will concentrate on the case $\ell = \mathbb{Z} \times \mathbb{C}$. We denote the resulting flavoured KLRW algebra by ${}^{\mathbb{Z}}\widetilde{\text{fT}}^\varphi$. Note that for any strand in a diagram in this algebra, the \mathbb{Z} -components of longitude must be the same at the top and bottom of the diagram. Thus, these strands are naturally labelled with elements of the product $\mathbb{Z} \times I$.

Given a $\mathbb{Z} \times \mathbb{C}$ -flavoured sequence $(\mathbf{i}, \mathbf{a}, <)$, we let $v_i^{(p)}$ be the number of $k \in \mathbb{C}$, such that $i_k = i$ and $a_k \in \{p\} \times \mathbb{C}$. These quantities will be unchanged from the top to the bottom of a diagram, as they correspond to the number of strands with label (p, i) . Moreover, the top and bottom of a diagram will differ by the action of $\prod_{i,p} \mathcal{S}_{v_i^{(p)}} \ltimes \mathbb{Z}^{v_i^{(p)}}$, which acts on the \mathbb{C} -component of the longitudes. Thus, it is natural to consider ${}^{\mathbb{Z}}\widetilde{\text{fT}}_{\mathcal{S}^{(p)}}^\varphi$, the subalgebra where the longitudes of the form $\{p\} \times \mathbb{C}$, give points in a fixed orbit $\mathcal{S}^{(p)} \subset \prod_i \mathbb{C}^{v_i^{(p)}}$.

We will also be interested in the usual (longitude values in \mathbb{C}) flavoured KLRW algebra $\widetilde{\text{fT}}_{\mathcal{S}^{(p)}}^\varphi$ corresponding to one fixed $\mathcal{S}^{(p)}$. The cases $p = 0$ and $p \neq 0$ play slightly different roles here, since only the former case keeps the Crawley-Boevey vertex; thus $\mathbf{w}^{(0)} = \mathbf{w}$ and $\mathbf{w}^{(p)} = 0$ otherwise.

We have a map

$$\bigotimes_{p \in \mathbb{Z}} \widetilde{\text{fT}}_{\mathcal{S}^{(p)}}^\varphi \rightarrow {}^{\mathbb{Z}}\widetilde{\text{fT}}_{\mathcal{S}^{(*)}}^\varphi, \tag{8.1}$$

given by replacing a longitude a appearing in a diagram in $\widetilde{\text{fT}}_{\mathcal{S}^{(p)}}^\varphi$ with (p, a) , and then horizontally composing the corresponding diagrams with p increasing from left to right.

Lemma 8.5. *The map (8.1) is an isomorphism of algebras.*

Proof. One can easily check that this map is injective by comparing polynomial representations; this is parallel to the argument of [Web17a, Corollary 4.15].

Thus, we need only show that it is surjective. To see this, note that at both the top and bottom of the diagram, strands are weakly ordered left to right by the first factor in their longitude. Thus, two strands where these factors are different must be in the same order at the top and bottom of the diagram. This implies that the resulting KLRW diagram can be isotoped to be the horizontal composition of diagrams only crossing strands with the same first factor. That is, the map is surjective as well. \square

Consider two sets ℓ, ℓ' as above, with actions of A, A' and a subset $R \subset \ell \times \ell'$; we assume that this satisfies two conditions on the set ${}_{xR} = \{y \in \ell' \mid (x, y) \in R\}$ and $R_y = \{x \in \ell \mid (x, y) \in R\}$:

1. The sets ${}_{xR}$ and R_y are closed under the equivalence relations \sim, \sim' .
2. The preorder \leq induces a total order on ${}_{xR}$ and R_y and on the set of equivalence classes in these sets: that is, if $x_1 \leq x_2$, then $x'_1 \leq x'_2$ whenever $x_1 \sim x'_1$ and $x_2 \sim x'_2$.

Definition 8.6. Let $\mathcal{F}(R)$ be the set of flavoured KLR diagrams equipped with a ℓ -flavoured sequence at the top and a ℓ' -flavoured sequence at the bottom, modulo the flavoured KLR relations (7.2a–7.21) in the case where the labels T at the top of the strands involved and labels B at the bottom of all strands involved satisfy $T \times B \subset R$, and the isotopy relations otherwise.

Lemma 8.7. *The vector space $\mathcal{F}(R)$ is a $\widetilde{\text{fT}}_\ell$ - $\widetilde{\text{fT}}_{\ell'}$ bimodule.*

Proof. Obviously, the composition of a diagram in $\mathcal{F}(R)$ with a flavoured KLRW diagram on the left or right gives a new flavoured KLRW diagram. Furthermore, this clearly preserves the relations in $\mathcal{F}(R)$, by locality. Thus, the only subtle point is why attaching a relation in $\widetilde{\text{fT}}_\ell$ gives a relation in $\mathcal{F}(R)$.

First, assume we have an ‘interesting’ relation in $\widetilde{\text{fT}}_\ell$, that is, one involving strands whose labels are equivalent under \sim . This attaches to two or three terminals at the top of the diagram, whose labels x_1, x_2, x_3 are all equivalent. The labels y_1, y_2, y_3 on the other end of these strands may not be equivalent under \sim' , but since R_{y_i} is closed under \sim , we have that $(x_i, y_j) \in R$, and so this is also a relation in $\mathcal{F}(R)$.

Now, assume we have a ‘boring’ relation in $\widetilde{\text{fT}}_\ell$, that is, at least one of the labels x_i is not equivalent to the others. As above, let y_j be the labels on the other end of these strands. If one of $(x_i, y_j) \notin R$, then the same boring relation holds in $\mathcal{F}(R)$. Thus, assume that all $(x_i, y_j) \in R$. Thus, $x_i \in R_{y_j}$ for all i and j . By assumption, this means that the elements $x_1 < x_2 < x_3$ are totally ordered, and we must have that x_1 or x_3 is not equivalent to x_2 , that is, we can’t have $x_1 \sim x_3 \not\sim x_2$. We thus can’t have a crossing between the corresponding strands, and thus the relation cannot appear in this case. \square

Remark 8.8. The bimodule $\widetilde{\text{FW}}$ from Section 7.4 is an example of such a bimodule $\mathcal{F}(R)$, where $\ell = \mathbb{C}$ and $\ell' = \mathbb{R}$ are as in the table above, and where $R = \{(x, y) \mid x \in \mathbb{Z}, y \in \mathbb{R}\}$.

We’ll principally be interested in this bimodule in the case where $\ell = \mathbb{C}$, and $\ell' = \mathbb{Z} \times \mathbb{C}$, and the relation $R = \{(y, (m, x)) \mid y - x \in \mathbb{Z}\}$. This defines a $\widetilde{\text{fT}}^\varphi$ - ${}^{\mathbb{Z}}\widetilde{\text{fT}}^\varphi$ bimodule \mathcal{F} given by the set of flavoured KLRW diagrams, with longitudes at the top given by elements of \mathbb{C} and at the bottom by elements of $\mathbb{Z} \times \mathbb{C}$. We will abuse notation, and use \mathcal{F} to denote the same bimodule, with the right action transferred by the homomorphism of (8.1) to one of $\bigotimes_{p \in \mathbb{Z}} \widetilde{\text{fT}}^\varphi_{\mathcal{S}^{(p)}}$. We can also define a ${}^{\mathbb{Z}}\widetilde{\text{fT}}^\varphi$ - $\widetilde{\text{fT}}^\varphi$ bimodule $co\mathcal{F}$ by swapping the role of top and bottom.

Definition 8.9. The functor of **restriction** associated to $\mathbf{v}^{(*)}$ is the functor

$$\text{Res} = \text{Hom}(\mathcal{F}, -) = co\mathcal{F} \otimes - : \widetilde{\text{fT}}^\varphi_{\mathcal{S}}\text{-mod} \rightarrow \bigotimes_{p \in \mathbb{Z}} \widetilde{\text{fT}}^\varphi_{\mathcal{S}^{(p)}}\text{-mod}.$$

The functor of **induction** associated to $\mathbf{v}^{(*)}$ is the left adjoint functor

$$\text{Ind} = \mathcal{F} \otimes - : \bigotimes_{p \in \mathbb{Z}} \widetilde{\text{fT}}^\varphi_{\mathcal{S}^{(p)}}\text{-mod} \rightarrow \widetilde{\text{fT}}^\varphi_{\mathcal{S}}\text{-mod},$$

and that of **coinduction** is the right adjoint $\text{Coind} = \text{Hom}(co\mathcal{F}, -)$.

While this bimodule is canonical, we can describe it in terms of an algebra homomorphism, at the price of making some noncanonical choices. Fix a finite subset $\mathcal{S}^{(p)} \subset \mathcal{S}^{(p)}$ and fix an integer $H \gg 0$. Consider the map $\sqcup_p \{p\} \times \mathcal{S}^{(p)} \rightarrow \mathbb{C}$ given by $(p, x) \mapsto Hp + x$. For H sufficiently large, this map is order preserving; note that this would never be the case on all of $\mathbb{Z} \times \mathbb{C}$, hence the need to choose finite subsets. With the above choices fixed, we have a homomorphism \wp to $\widetilde{\text{fT}}^\varphi_{\mathcal{S}}$ from the subalgebra of ${}^{\mathbb{Z}}\widetilde{\text{fT}}^\varphi$, where we fix all longitudes to live in $\sqcup_p \{p\} \times \mathcal{S}^{(p)}$, and \mathcal{F} matches the induced

bimodule of this homomorphism. We can choose $S^{(p)}$ so that this subalgebra is Morita equivalent to ${}^{\mathbb{Z}}\widetilde{\text{fl}}_{\mathbf{v}}^{\varphi}$. Thus, using this Morita equivalence, we can define the functors of Definition 8.9 as the usual induction/restriction/coinduction functors of a ring homomorphism.

8.2. Comparison with other constructions

Let us restrict to the integral case throughout this section. Note that, as usual, we can reduce the general case to the integral case using Lemma 7.21.

Lemma 8.10. *Under the quotient functor of Theorem 7.23, the induction/restriction functors for flavoured KLRW algebras match the induction/restriction functors for weighted KLRW algebras defined in [Web19b, Definition 2.17]:*

$$\begin{array}{ccc}
 \widetilde{\text{fl}}_{\mathbf{v}}^{\varphi}\text{-mod} & \begin{array}{c} \xleftarrow{\text{Res}} \\ \xrightarrow{\text{Ind}} \end{array} & \bigotimes_{k \in \mathbb{Z}} \widetilde{\text{fl}}_{\mathbf{v}^{(k)}}^{\varphi}\text{-mod} \\
 \text{FW} \otimes - \uparrow & & \uparrow \mathbb{Z}\text{FW} \otimes - \\
 \widetilde{\text{T}}_{\mathbf{v}}^{\vartheta}\text{-mod} & \begin{array}{c} \xleftarrow{\text{Res}} \\ \xrightarrow{\text{Ind}} \end{array} & \bigotimes_{k \in \mathbb{Z}} \widetilde{\text{T}}_{\mathbf{v}^{(k)}}^{\vartheta}\text{-mod}
 \end{array}$$

In fact, we could define the restriction functors for weighted KLRW algebras using a bimodule $\mathcal{S}(R)$ from Definition 8.6: take $\ell = \mathbb{R}$ which gives weighted KLRW algebras, take $\ell' = \mathbb{Z} \times \mathbb{R}$ equipped with lexicographic order and the equivalence relation $(m, x) \sim (m', x')$ iff $m = m'$, and take the full relation $R = \mathbb{R} \times (\mathbb{Z} \times \mathbb{R})$.

Let us consider the interaction between these functors and steadied quotients. Note that an idempotent is unsteady if and only if it is in the image of the ring homomorphism discussed above with $\mathbf{v}^{(p)} \neq 0$ for some positive integer p . In particular, any module M over $\widetilde{\text{fl}}_{\mathbf{v}}^{\varphi}$ that factors through the steadied quotient $\text{fl}_{\mathbf{v}}^{\varphi}$ is killed by Res if $\mathbf{v}^{(p)} \neq 0$ for some positive integer p .

We can think about this a bit more systematically by considering the tensor product $\widetilde{\text{fl}}_{\mathbf{v}}^{\varphi} \otimes \mathcal{S}$. The right action on this tensor product obviously factors through the steadied quotient of ${}^{\mathbb{Z}}\widetilde{\text{fl}}_{\mathbf{v}}^{\varphi}$ (repeating Definition 7.17 verbatim). Since all strands with labels (p, i) with $p > 0$ form a group that unsteadies the sequence, we must have no such strands in the steadied quotient. Similarly, if the strands with label $(0, i)$ considered on their own give an unsteady sequence, the same is true of the sequence as a whole. From these observations, it's easy to check that:

Lemma 8.11. *Given $\mathbf{v}^{(0)}, \mathbf{v}^{(-1)}, \dots, \mathbf{v}^{(-m)}$ with $\mathbf{v} = \mathbf{v}^{(-m)} + \dots + \mathbf{v}^{(0)}$, we have an isomorphism of algebras between the steadied quotients of the algebras ${}^{\mathbb{Z}}\widetilde{\text{fl}}_{\mathbf{v}}^{\varphi}$ and $\widetilde{\text{fl}}_{\mathbf{v}^{(-m)}}^{\varphi} \otimes \widetilde{\text{fl}}_{\mathbf{v}^{(-m+1)}}^{\varphi} \otimes \dots \otimes \widetilde{\text{fl}}_{\mathbf{v}^{(0)}}^{\varphi}$. Thus, in this case, we obtain a well-defined functor*

$$\text{Res} = \text{Hom}(\mathcal{S}, -) = \text{co}\mathcal{S} \otimes - : \widetilde{\text{fl}}_{\mathbf{v}}^{\varphi}\text{-mod} \rightarrow \widetilde{\text{fl}}_{\mathbf{v}^{(-m)}}^{\varphi} \otimes \widetilde{\text{fl}}_{\mathbf{v}^{(-m+1)}}^{\varphi} \otimes \dots \otimes \widetilde{\text{fl}}_{\mathbf{v}^{(0)}}^{\varphi}\text{-mod}.$$

Note that this restriction functor commutes with inflation from steadied quotients to the full algebra. The same is not true of its left and right adjoints.

8.3. Categorical actions

In this subsection, we assume that Γ has no edge loops, and continue to only consider the integral case.

We now focus on the functor from Lemma 8.11 in the case where $m = 1$, and $\mathbf{v}^{(-1)}$ is a multiple of a unit vector, that is, supported on a single vertex i , with some multiplicity k . Let NH_k be the nilHecke algebra on k strands; this is the algebra given by KLR diagrams (without red strands, ghosts or longitudes) with k strands that have the same label, satisfying the relations (7.2a–7.2c; see [KL09, Section 2.2(3)].

Lemma 8.12. For any idempotent $e((i, \dots, i), \mathbf{a}, <)$, we have an isomorphism

$$e((i, \dots, i), \mathbf{a}, <) \widetilde{\text{fl}}_{\mathbf{v}^{(-1)}}^\varphi e((i, \dots, i), \mathbf{a}, <) \cong \text{NH}_k,$$

and this induces a Morita equivalence between $\widetilde{\text{fl}}_{\mathbf{v}^{(-1)}}^\varphi$ and NH_k .

This result depends on the lack of edge loops; if Γ were to have an edge loop, $\widetilde{\text{fl}}_{\mathbf{v}^{(-1)}}^\varphi$ would be much more complicated.

Proof. In this case, all strands have the same label and there are no ghosts or red strands, so only relations (7.2a–7.2a) are relevant. Thus, simply forgetting labels and longitudes gives the desired map to NH_k , and adding them back its inverse. For any pair $((i, \dots, i), \mathbf{a}, <)$ and $((i, \dots, i), \mathbf{a}', <')$, we can simply label the identity diagram in NH_k of k vertical lines with \mathbf{a} in the order $<$ at the top and \mathbf{a}' in the order $<'$ at the bottom, and obtain an isomorphism between these idempotents, showing that $e((i, \dots, i), \mathbf{a}, <)$ generates $\widetilde{\text{fl}}_{\mathbf{v}^{(-1)}}^\varphi$ as a two-sided ideal. Thus, we have the desired Morita equivalence. \square

The core result that makes all of higher representation theory work is that NH_k is isomorphic to the rank $n!$ matrix algebra over the symmetric polynomials Sym_k in k -variables.

Thus, the ring $\text{NH}_k \otimes_{-\text{fl}}_{\mathbf{v}^{(0)}}^\varphi$ is Morita equivalent to $\text{Sym}_k \otimes \text{fl}_{\mathbf{v}^{(0)}}^\varphi$.

Definition 8.13. The divided power functor $\mathcal{E}_i^{(k)} : \text{fl}_{\mathbf{v}}^\varphi\text{-mod} \rightarrow \text{fl}_{\mathbf{v}^{(0)}}^\varphi\text{-mod}$ is the composition of the restriction functor $\text{fl}_{\mathbf{v}}^\varphi\text{-mod} \rightarrow \text{NH}_k \otimes \text{fl}_{\mathbf{v}^{(0)}}^\varphi\text{-mod}$, followed by this Morita equivalence and forgetting the action of Sym_k . Let $\mathcal{F}_i^{(k)}$ be the left adjoint of $\mathcal{E}_i^{(k)}$.

Exactly as in Lemma 8.10, these functors match under the quotient functor of Theorem 7.23 with the categorical Lie algebra action of [Web19b, Theorem3.1]. Thus, we have that:

Proposition 8.14. The functors \mathcal{E}_i and \mathcal{F}_i for $i \in I$ give a categorical \mathfrak{g}_Γ -action, sending the weight $\mu = \sum w_i \varpi_i - v_i \alpha_i$ to the category $\text{fl}_{\mathbf{v}}^\varphi\text{-mod}$.

It is natural to ask which representation of \mathfrak{g}_Γ is categorified by Proposition 8.14. In the papers [KTW⁺19a, KTW⁺19b], we showed that for bipartite simply-laced Kac-Moody types, this representation is described by the product monomial crystal. In [Gib21], it is shown that in finite type A, this representation can be identified with a generalized Schur module, and can be described via a generalized Demazure module in all finite types. In general, we will now explain that this representation always surjects onto the irreducible representation with highest weight $\lambda = \sum_i w_i \varpi_i$, and can sometime be identified with a tensor product of fundamental representations.

We always have an equivariant map $\bigoplus \text{fl}_{\mathbf{v}}^\varphi\text{-mod} \rightarrow \bigoplus R_{\mathbf{v}}^w\text{-mod}$ induced by the quotient functor $M \mapsto e_H M$ (see Proposition 7.26), and thus an equivariant map $K_{\mathbb{C}}(\bigoplus_{\mathbf{v}} \text{fl}_{\mathbf{v}}^\varphi\text{-mod}) \rightarrow K_{\mathbb{C}}(\bigoplus_{\mathbf{v}} R_{\mathbf{v}}^w\text{-mod})$. This latter Grothendieck group is an irreducible representation of \mathfrak{g}_Γ with highest weight λ by a special case of [Web17a, Theorem B]. This map is typically not an isomorphism but can be in extremely degenerate cases, such as when all $\varphi_e = 0$.

For the next result, we will need to assume that $\varphi_e = 0$ for all old edges. For Γ without a cycle, all cases can be reduced to this one by adding a 1-coboundary, using Lemma 7.21(2).

Lemma 8.15. Let $\lambda_1, \dots, \lambda_n$ be fundamental weights with $\sum \lambda_k = \lambda$. Suppose that $\varphi_e = 0$ for all old edges and $|\varphi_e - \varphi_{e'}| > \sum v_i$ for all new edges e, e' . Then, $K_{\mathbb{C}}(\bigoplus \text{fl}_{\mathbf{v}}^\varphi\text{-mod}) \cong V(\lambda_1) \otimes \dots \otimes V(\lambda_n)$ as \mathfrak{g}_Γ -modules.

Proof. By [Web19b, Theorem 3.6], in this case, the weighted KLRW algebra is the KLRW algebra T^λ , so $\text{fl}_{\mathbf{v}}^\varphi\text{-mod}$ is a quotient of $T^\lambda\text{-mod}$. The kernel of this quotient is the modules killed by all idempotents which cannot be realized as a flavoured sequence.

Consider an idempotent in T^λ ; the potential difficulty of realizing this with a flavoured sequence is that if we want to make sure that two strands are in the correct order, they may need to have different

longitudes, and there might not be enough different longitudes between the values φ_e and $\varphi_{e'}$ for two new edges.

However, the condition $|\varphi_e - \varphi_{e'}| > \sum v_i$ guarantees that there are enough different possible longitudes between any two of these values to accommodate all corporeal strands, all with different longitudes, and thus having any order we desire. This shows we have a Morita equivalence and completes the proof, since $K_{\mathbb{C}}(T^\lambda) \cong V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)$ by [Web17a, Theorem 4.38]. \square

Remark 8.16. As we've done several times now, we can consider the nonintegral case using the usual trick of Lemma 7.21. In this case, we'll apply the theorems above for the quiver $\tilde{\Gamma}$. In particular, it might be that Γ has edge loops and $\tilde{\Gamma}$ does not. For example, when Γ is the Jordan quiver and the flavour on its edge is not integral, $\tilde{\Gamma}$ will have no edge loops; it will be a finite subset of a union of cycles or infinite linear quivers, depending on whether the label is rational or irrational. This case has been extensively considered in [Web17b]; in particular, its Grothendieck group is calculated in [Web17b, Theorem B] as a higher level Fock space.

9. Parabolic restriction and flavoured KLRW algebras

In this section, we'll apply the theory developed in Section 5 to the case of a quiver gauge theory.

We begin by explaining how flavoured KLRW algebras appear in relation to the Coulomb branches of quiver gauge theories. In Theorem 9.5, we show that $\mathcal{A}(\mathfrak{v}, \mathfrak{w}) - \Gamma\Pi_{\mathcal{G}}$ is equivalent to a category of modules over a flavoured KLRW algebra, and we show in Theorem 9.12 that this equivalence is compatible with the induction and restriction functors. In Section 9.3, we define (divided powers of) restriction and induction functors on category \mathcal{O} over Coulomb branch algebras, and show in Theorem 9.17 that these induce categorical Lie algebra action, in the quiver case.

9.1. Relation between Coulomb branches and flavoured KLRW algebras

As before, we fix a quiver Γ , dimension vectors $\mathfrak{w}, \mathfrak{v} \in \mathbb{Z}_{\geq}^I$ and write $\Gamma^{\mathfrak{w}}$ for the Crawley-Boevey quiver. Recall that we are considering a quiver gauge theory with

$$G = \prod GL(v_i) \quad F = (\mathbb{C}^\times)^{E(\Gamma^{\mathfrak{w}})} \quad N = \bigoplus_{e \in E(\Gamma^{\mathfrak{w}})} \text{Hom}(\mathbb{C}^{v_{t(e)}}, \mathbb{C}^{v_{h(e)}}).$$

Remark 9.1. Note that there is some redundancy here. The map from \tilde{G} into $GL(N)$ is usually not injective. Thus, any flavours that agree up to a coweight into the kernel give isomorphic specializations of $\mathcal{A}(G, N)$; these are precisely the weightings which are cohomologous when thought of as 1-cocycles.

Similarly, any coweights conjugate in the normalizer of G in $GL(N)$ (even if they are not conjugate in \tilde{G}) give isomorphic specializations of the Coulomb branch. For example, we can permute the flavours on edges joining the same pair of vertices.

A point $\varphi \in \mathfrak{f}$ is given by a map $E(\Gamma^{\mathfrak{w}}) \rightarrow \mathbb{C}$. This is the same data as a flavour (Definition 7.7). For the remainder of this section, we will fix such a flavour $\varphi: Z \rightarrow \mathbb{C}$ and study only the Gelfand-Tsetlin modules for $\mathcal{A}(G, N)$, where the centre Z acts by this character. Of course, this is equivalent to studying modules over the quotient algebra $\mathcal{A}_\varphi := \mathcal{A}_\varphi(G, N)$.

The image of the Gelfand-Tsetlin algebra in \mathcal{A}_φ is $\mathbb{C}[t + \varphi]^W$. Since φ is fixed and $\tilde{\mathfrak{t}} = \mathfrak{t} \oplus \mathfrak{f}$, we will identify $\mathfrak{t} + \varphi = \mathfrak{t}$. Thus, a Gelfand-Tsetlin weight for \mathcal{A}_φ will be given by a point $\gamma \in \mathfrak{t}/W = \prod_i \mathbb{C}^{v_i}/S_{v_i}$. As in Section 7.2, we will think of γ as a collection (γ_i) of size v_i multisets. By Definition 7.15, there is a corresponding idempotent $e(\gamma)$, where γ_i gives the longitudes a_k with label $i_k = i$.

Given $\gamma \in \mathfrak{t}/W$, a lift $\lambda \in \mathfrak{t}$ corresponds to choosing an ordering on each multiset. Let $W^\lambda \subset W$ be the stabilizer of λ ; this is simply the product of symmetric groups corresponding to repeated elements in each γ_i . Thus, $W^\lambda \neq \{1\}$ if and only if there is repetition in one of the multisets γ_i . Equivalently, we have consecutive corporeal elements of CGR in the flavoured sequence coming from γ with the same vertex

and longitude. As in [KTW⁺19b, Section 5.2], crossings of consecutive strands with the same longitude and label generate a copy of the nilHecke algebra of W^λ in $e(\gamma)\widetilde{f}\Gamma e(\gamma)$. We let $e'(\gamma)$ be a primitive idempotent in this nilHecke algebra; for concreteness, we can take this to be the projection to W^λ .

Consider the flavoured KLRW algebra $\widetilde{f}\Gamma^\varphi(\Gamma^w)$ for the flavour φ . Let $\widetilde{f}\Gamma^\varphi$ -wmod be the category of locally finite-dimensional weakly gradable modules over $\widetilde{f}\Gamma^\varphi$, that is, the modules with a filtration whose subquotients are gradable and where $e(\mathbf{i}, \langle \rangle)M$ is finite-dimensional for all such idempotents. We can also characterize these modules topologically: $\widetilde{f}\Gamma^\varphi$ carries its **grading topology**, the coarsest topology where the elements of degree $\geq k$ for each k define a basis of neighbourhoods of the identity.

Lemma 9.2. *A locally finite-dimensional $\widetilde{f}\Gamma^\varphi$ module M is weakly gradable if and only if as a discrete topological module, it carries a continuous action of the completion $\widehat{\widetilde{f}\Gamma^\varphi}$ in the grading topology.*

Proof. We use several times here that $\widetilde{f}\Gamma^\varphi$ only has finitely many idempotents up to isomorphism. In particular, this means that a locally finite-dimensional weakly gradable module is killed by all elements of sufficiently high degree, so indeed, it carries an action of this completion.

This also means that the category of locally finite modules over this ring is Artinian and the quotient by the ideal generated by elements of degree $\geq k$ is Morita equivalent to a finite-dimensional graded algebra. Thus, if M is a module with such a continuous action, it contains a simple submodule, which is gradable by the Morita equivalence mentioned above. Induction shows that M is weakly gradable. \square

Definition 9.3. Let $\widehat{\mathcal{A}}_\varphi$ be the category with objects \mathfrak{t}/W and morphisms given by

$$\text{Hom}_{\widehat{\mathcal{A}}_\varphi}(\gamma, \gamma') = \text{Hom}(W_\gamma, W_{\gamma'}),$$

where W_γ denotes the weight functor on the category of \mathcal{A}_φ -modules.

Let $\widehat{\mathcal{T}}_\varphi$ be the category with objects \mathfrak{t}/W and morphisms given by

$$\text{Hom}_{\widehat{\mathcal{T}}_\varphi}(\gamma, \gamma') = e'(\gamma') \cdot \widehat{\widetilde{f}\Gamma^\varphi} \cdot e'(\gamma).$$

Both of these categories have the property that Hom between γ and γ' is trivial unless γ and γ' are in the image of a single \widehat{W} orbit in \mathfrak{t} .

Thus, we can write $\widehat{\mathcal{A}}_\varphi$ as a direct sum of subcategories $\widehat{\mathcal{A}}_\mathcal{S}$, where \mathcal{S} ranges over \widehat{W} -orbits in \mathfrak{t} , and similarly with $\widehat{\mathcal{T}}_\varphi \subset \widehat{\mathcal{A}}_\varphi$.

Recall that in Definition 6.1, we defined a category $\widehat{\mathcal{A}}_\mathbb{Z}(G, N)$ with objects $\mathfrak{t}_\mathbb{Z}/W$. We define $\widehat{\mathcal{A}}_{\mathbb{Z}, \varphi}(G, N)$ to be the full subcategory of this category whose objects are the subset $(\mathfrak{t}_\mathbb{Z} + \varphi)/W$. Note that if φ is integral and we choose $\mathcal{S} = \mathfrak{t}_\mathbb{Z} + \varphi$, then our two definitions coincide: $\widehat{\mathcal{A}}_\mathcal{S} = \widehat{\mathcal{A}}_{\mathbb{Z}, \varphi}(G, N)$.

In fact, understanding this integral case gives us all the tools we need to understand all blocks. Recall the construction of the quiver $\widetilde{\Gamma}$ in Definition 7.19, together with its dimension vectors $\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}$ and flavour φ' used in Lemma 7.21. Moreover, we let G', N' be the gauge group and matter representation associated to the quiver $\widetilde{\Gamma}$, with dimension vectors $\widetilde{\mathbf{v}}, \widetilde{\mathbf{w}}$.

By [Weba, Corollary 4.7], we have the following.

Lemma 9.4. *The subcategory $\widehat{\mathcal{A}}_\mathcal{S}$ is equivalent to $\widehat{\mathcal{A}}_{\mathbb{Z}, \varphi'}(G', N')$.*

See [SW, Section 4.2.3] for a discussion of how this result can be applied to understand all blocks of Gelfand-Tsetlin modules for $U(\mathfrak{gl}_3)$.

The following result is a direct generalization of [KTW⁺19b, Theorem 5.2]. The proof of this theorem will be given below.

Theorem 9.5. *We have an equivalence $\widehat{\mathcal{T}}_\varphi \cong \widehat{\mathcal{A}}_\varphi$. In particular, for any set $S \subset \mathfrak{t}$, there is an isomorphism*

$$F(S) \cong \widehat{\widetilde{f}\Gamma}_S^\varphi. \tag{9.1}$$

From the theorem and Proposition 4.5, we immediately conclude the following:

Corollary 9.6. *For any orbit $\mathcal{S} \subset \mathfrak{t}$, there is an equivalence*

$$\mathcal{W} : \mathcal{A}(G, N) - \Gamma \Pi_{\mathcal{S}} \rightarrow \widetilde{\text{fl}}_{\mathcal{S}}^{\varphi} - \text{wmod},$$

such that for any $M \in \mathcal{A}(G, N) - \Gamma \Pi_{\mathcal{S}}$ and any $\gamma \in \overline{\mathcal{S}}$, we have

$$\mathcal{W}_{\gamma}(M) = e'(\gamma)\mathcal{W}(M).$$

Before beginning the proof of this theorem, we will examine the spaces Y_{γ}, X_{λ} (from Section 6.1) in the quiver case. Assume for now that φ is integral; Theorem 9.5 holds for all φ , but we will reduce to the integral case in the proof, and for simplicity, we only state Lemmata 9.8 and 9.9 in the integral case. For any $\gamma \in \prod_i \mathbb{Z}^{v_i}/S_{v_i}$, let $|\gamma_i^{\leq k}| = |\gamma_i \cap (-\infty, k]|$, that is, the number of elements in the multiset γ_i which are at most k .

Definition 9.7. A partial flag of type γ is a \mathbb{Z} -indexed sequence of nested subspaces

$$\cdots \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq \bigoplus_{i \in I \sqcup \{\infty\}} \mathbb{C}^{v_i},$$

such that each F_k is compatible with the decomposition $\bigoplus \mathbb{C}^{v_i}$, and for all $k \in \mathbb{Z}, i \in I$, we have

$$\dim F_k \cap \mathbb{C}^{v_i} = |\gamma_i^{\leq k}|.$$

Here, we include ∞ as the one-dimensional Crawley-Boevey vertex, with $\mathbb{C}^{v_{\infty}} = \mathbb{C}_{\infty}$. We assume that $\gamma_{\infty} = \{0\}$, which forces

$$\dim F_k \cap \mathbb{C}_{\infty} = \begin{cases} 1 & \text{if } k \geq 0 \\ 0 & \text{if } k < 0. \end{cases}$$

Each partial flag of type γ gives rise to an ordinary partial flag (indexed by dimension) in each \mathbb{C}^{v_i} .

A careful examination of the definitions of X_{λ}, Y_{γ} leads to the following result.

Lemma 9.8. *Consider $\gamma \in \prod_i \mathbb{Z}^{v_i}/S_{v_i}$, and let $\lambda \in \prod_i \mathbb{Z}^{v_i}$ a lift of γ .*

1. *The partial flag variety G/P_{λ} is isomorphic to the space of partial flags of type γ .*
2. *Applying the second description of (6.1):*

$$Y_{\gamma} = \{(F_{\bullet}, n) \in G/P_{\lambda} \times N \mid n_e(F_k) \subset F_{k+\varphi_e} \text{ for all } e \in E(E^{\mathbf{w}}), k \in \mathbb{Z}\}.$$

3. *The fibre of $X_{\lambda} \rightarrow Y_{\gamma}$ over (F_{\bullet}, n) is given by choosing full flags in each \mathbb{C}^{v_i} , refining the partial flags coming from F_{\bullet} .*

From Lemma 7.9, we can associate a φ -flavoured sequence to each γ , where the entries of each multiset γ_i give the longitudes a_k with $i_k = i$. Then, given the flavoured sequence, we can construct a loading ℓ as explained in Section 7.4.

To each loading ℓ , in [Web19b, Definition 4.2], the second author defined a space X_{ℓ} of ℓ -loaded flags and compatible representations; these assign a subspace F'_a to each real number a whose dimension vector is given by summing the labels on real numbers $\leq a$ under the loading. Assume that ℓ is the loading $\ell(\gamma)$ defined in Section 7.4, attached to the idempotent $e(\gamma)$. Given an element of X_{ℓ} , we have subspaces $F_k = F'_{k+1/2}$ which define a partial flag of type γ since there are precisely $|\gamma_i^{\leq k}|$ appearances of i in this loading on real numbers $\leq a + 1/2$, which is an element of Y_{γ} by Lemma 7.22. Since considering the spaces $F'_{k+\epsilon m}$ for different values of m gives a refinement of this to a complete flag, we have that:

Lemma 9.9. *There is an isomorphism $X_{\ell} \cong X_{\lambda}$.*

Proof of Theorem 9.5. Let us fix a single orbit \mathcal{S} and only consider the construction of the equivalence of this block. By Lemmata 7.21 and 9.4, we can reduce to the case where the flavour φ is integral and $\mathcal{S} = \mathfrak{t}_{\mathbb{Z}} + \varphi$.

By Corollary 6.7, this reduces to showing that for all $\gamma, \gamma' \in \mathfrak{t}_{\mathbb{Z}}$,

$$e'(\gamma') \widehat{\text{fT}}^{\varphi} e'(\gamma) \cong \widehat{H}^G(\gamma \mathbb{Y}_{\gamma'}). \tag{9.2}$$

By (6.9) applied with $G = L$, it suffices to establish an isomorphism

$$e(\gamma') \widehat{\text{fT}}^{\varphi} e(\gamma) \cong \widehat{H}^G(\lambda \mathbb{X}_{\lambda'}), \tag{9.3}$$

where λ, λ' are antidominant lifts of γ, γ' .

Since by Theorem 7.23, the tensor product $\widehat{\text{FW}} \otimes_{\widetilde{\text{FV}}^{\varphi}} -$ is a quotient functor, we have that $\widehat{\text{FW}}$ is projective as a left module, and $\text{Hom}(\widehat{\text{FW}}, \widetilde{\text{T}}^{\varphi} e(\ell)) \cong \widetilde{\text{fT}}^{\varphi} e(\gamma)$, where ℓ is the loading associated to $e(\gamma)$ in the weighted KLRW algebra. This shows that $e(\gamma') \widetilde{\text{fT}}^{\varphi} e(\gamma) \cong e(\ell') \widetilde{\text{T}}^{\varphi} e(\ell)$ since

$$e(\ell') \widetilde{\text{T}}^{\varphi} e(\ell) \cong \text{Hom}(\widehat{\text{FW}} \otimes_{\widetilde{\text{FV}}^{\varphi}} \widetilde{\text{fT}}^{\varphi} e(\gamma), \widetilde{\text{T}}^{\varphi} e(\ell)) \cong \text{Hom}(\widetilde{\text{fT}}^{\varphi} e(\gamma), \widetilde{\text{fT}}^{\varphi} e(\gamma')) \cong e(\gamma') \widetilde{\text{fT}}^{\varphi} e(\gamma).$$

On the other hand, [Web19b, Theorem 4.5] describes this part of the weighted KLRW algebra using equivariant homology of a fibre product. The isomorphism for flavoured algebras can also be written directly using the same philosophy:

1. diagrams with no dots, no double crossings and no pair of strands with a same label and longitudes that differ by \mathbb{Z} crossing are sent to the homology class of preimage of the diagonal under the map $\lambda \mathbb{X}_{\lambda'} \rightarrow G/B \times G/B$.
2. dots are sent to the Chern classes of tautological line bundles on this preimage in $\lambda \mathbb{X}_{\lambda}$.
3. diagrams with a single crossing of a pair of strands with a same label and longitudes that differ by \mathbb{Z} and no other crossings are sent to the preimage of the diagonal in $G/P \times G/P$, where P is the parabolic where we add in the simple root corresponding to the crossing.

By Lemma 9.9, this fibre product is isomorphic to $\lambda \mathbb{X}_{\lambda'}$. All of these isomorphisms are compatible with composition and convolution in homology, so this defines the desired equivalence of categories. \square

9.2. Functors in the quiver case

Fix a coweight $\xi : \mathbb{C}^{\times} \rightarrow T$ as in Section 3.2. We will now see how the functors *res*, *ind* and *coind*, as defined in Section 5, interact with the equivalence of Theorem 9.5.

As discussed in Section 3.2, a choice of ξ gives us dimension vectors $\mathbf{v}^{(p)}$ for $p \in \mathbb{Z}$. Let $\gamma \in \mathfrak{t}/W = \prod_i \mathbb{C}^{v_i}/S_{v_i}$, as in the previous section. Choosing $\nu \in \mathfrak{t}/W_L = \prod_p \prod_i \mathbb{C}^{v_i^{(p)}}/S_{v_i^{(p)}}$ lifting γ means dividing each multiset γ_i into multisets $\nu_i^{(p)}$ of sizes $v_i^{(p)}$. With this in mind, Definition 5.6 translates into the following statement.

Lemma 9.10. *ν is ξ -negative if and only if*

1. *for all $p, q \in \mathbb{Z}, e \in \Gamma^{\mathbf{w}}, y \in \nu_{\mathfrak{t}(e)}^{(p)}, z \in \nu_{\mathfrak{h}(e)}^{(q)}$, we have*

$$\begin{aligned} &\text{if } p < q, \text{ then } \varphi_e + z - y \notin \mathbb{Z}_{>0} \\ &\text{and if } p > q, \text{ then } \varphi_e + z - y \notin \mathbb{Z}_{\leq 0} \end{aligned}$$

2. *for all $i \in I$ and $p \neq q \in \mathbb{Z}$, if $y \in \nu_i^{(p)}, z \in \nu_i^{(q)}$, then $y \neq z$*

(here as usual, we adopt the convention that $\nu_{\infty}^p = \{0\}$, if $p = 0$, and is empty otherwise).

We now fix an orbit $\mathcal{S} \subset \mathfrak{t} + \varphi$ for \widehat{W} , and $\mathcal{S}^L \subset \mathcal{S}$ an orbit of \widehat{W}_L . Since fixing \mathcal{S} requires choosing the multisets of the fractional parts in $\prod(\mathbb{C}/\mathbb{Z})^{v_i}/S_{v_i}$, \mathcal{S}^L corresponds to a division of these fractional parts into submultisets of the correct size. If all $\gamma_i \in \mathbb{Z}^{v_i}/S_{v_i}$, then we have $\mathcal{S} = \mathcal{S}^L$.

The orbit \mathcal{S}^L is the product of orbits $\mathcal{S}^{(p)}$ of the extended affine Weyl groups for the smaller gauge groups $L_p = \prod_i GL(\mathbb{C}^{v_i^{(p)}})$.

Now, we choose a set $S \subset \mathcal{S}$ whose corresponding flavoured sequences give a transversal of the equivalence classes. This must be a complete set in the sense of Definition 4.6 since if γ and γ' give equivalent flavoured sequences, by Theorem 9.5, the straight line diagram between them gives an isomorphism $\mathcal{W}_\gamma \cong \mathcal{W}_{\gamma'}$ as functors.

We can choose similar sets $S^{(p)} \subset \mathcal{S}^{(p)}$; we will assume that these are chosen so that any point in the product $\prod_{p \in \mathbb{Z}} S^{(p)}$, interpreted as an element of \mathfrak{t}/W_L , is ξ -negative. To see that this ξ -negativity property can be achieved, suppose that $S^{(p)}$ is to an arbitrary complete set of $\mathcal{S}^{(p)}$. Fix an integer $H \gg 0$. Then, modify $S^{(p)}$ by adding pH to all the entries in each multiset $\gamma_i^{(p)} \in S^{(p)}$ for all p . The resulting elements of $\prod_{p \in \mathbb{Z}} S^{(p)}$ will then be ξ -negative.

We define $S^L = \prod_{p \in \mathbb{Z}} S^{(p)}$, and as discussed before, we can assume that $S^L \subset S$.

As discussed in Section 3.2, we have $\mathcal{A}(L, N) = \bigotimes_p \mathcal{A}(L^{(p)}, N^{(p)})$. This leads to an isomorphism

$$F^L(S^L) = \bigotimes_{p \in \mathbb{Z}} F^{L^{(p)}}(S^{(p)}).$$

Combining this isomorphism with those from Corollary 9.6 and Lemma 8.5, we obtain,

$$F^L(S^L) \cong \bigotimes_{p \in \mathbb{Z}} F^{L^{(p)}}(S^{(p)}) \cong \bigotimes_{p \in \mathbb{Z}} \widetilde{f\Gamma}_{S^{(p)}}^\varphi \cong \mathbb{Z} \widetilde{f\Gamma}_{S^{(*)}}^\varphi. \tag{9.4}$$

Recall that in (5.3), we defined a $F(S) - F^L(S^L)$ bimodule $I(S^L, S)$. Also recall that in Section 8.1, we defined a $\widetilde{f\Gamma}_S^\varphi - \mathbb{Z} \widetilde{f\Gamma}_{S^{(*)}}^\varphi$ -bimodule \mathcal{F} . By completing with respect to the grading, we obtain a $\widehat{f\Gamma}_S^\varphi - \mathbb{Z} \widehat{f\Gamma}_{S^{(*)}}^\varphi$ -bimodule $\widehat{\mathcal{F}}$. Similarly, we have bimodules $I(S, S^L)$ and $\widehat{co}\mathcal{F}$ for these algebras in the opposite orders.

Lemma 9.11. *Under the isomorphisms (9.1) and (9.4), the bimodule $I(S^L, S)$ corresponds to the bimodule $\widehat{\mathcal{F}}$, and $I(S, S^L)$ to $\widehat{co}\mathcal{F}$.*

Proof. By (5.3), the bimodule $I(S^L, S)$ is given by the sum of Homs between weight functors

$$I(S^L, S) \cong \bigoplus_{\substack{\gamma' \in \widehat{S} \\ \nu \in S^L}} \text{Hom}(\mathcal{W}_\nu^L \circ \text{res}, \mathcal{W}_{\gamma'}).$$

For each $\nu \in S^L, \gamma' \in S$, let γ be the image of ν in \mathfrak{t}/W . Using the isomorphism Y' defined in (6.14), we can define an isomorphism $\text{Hom}(\mathcal{W}_\nu^L \circ \text{res}, \mathcal{W}_{\gamma'}) \cong \text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'})$.

Theorem 6.12 shows that the induced isomorphism of vector spaces

$$\text{Hom}(\mathcal{W}_\nu^L \circ \text{res}, \mathcal{W}_{\gamma'}) \cong \widehat{H}^G({}_\gamma Y_{\gamma'}) \quad I(S^L, S) \cong \bigoplus_{\substack{\gamma' \in S \\ \nu \in S^L}} \widehat{H}^G({}_\gamma Y_{\gamma'})$$

is an isomorphism of bimodules, where the left actions are intertwined by the functor E_G , and the right action by E_L and the saturation map (6.11).

By Theorem 9.5, we also have an isomorphism $\text{Hom}(\mathcal{W}_\gamma, \mathcal{W}_{\gamma'}) \cong e'(\gamma') \cdot \widehat{f\Gamma}^\varphi \cdot e'(\gamma)$. At the start of the proof, we chose a preimage ν for γ ; since $S^L = \prod_{p \in \mathbb{Z}} S^{(p)}$, we can let $\nu^{(p)} \in S^{(p)}$ be the projection of ν . By Lemma 7.9, there exists a flavoured sequence with $\nu^{(p)}$ as the set of longitudes. Concatenating these flavoured sequences together gives us a $\mathbb{Z} \times \mathbb{C}$ flavoured sequence which matches the sequence for

$e(\gamma)$ after projecting to the second factor. For each diagram in $e'(\gamma') \cdot \widehat{\text{fT}}^\varphi \cdot e'(\gamma)$, we can consider the same diagram with this sequence placed at the bottom and obtain an element of $\widehat{\mathcal{F}}$.

By definition, the resulting diagram is an element of $e(\gamma') \cdot \widehat{\mathcal{F}} \cdot e^{\mathbb{Z}}(\nu)$, where $e^{\mathbb{Z}}(\nu)$ is the corresponding idempotent in $\widehat{\text{fT}}^\varphi$. In fact, this operation on diagrams defines an isomorphism of vector spaces

$$e(\gamma') \cdot \widehat{\mathcal{F}} \cdot e^{\mathbb{Z}}(\nu) \cong e(\gamma') \cdot \widehat{\text{fT}}^\varphi \cdot e(\gamma),$$

where the map is simply projecting the longitudes in $\mathbb{Z} \times \mathbb{C}$ at the bottom of the diagram to their second component, and thus an isomorphism of vector spaces $F(\mathbb{S}^L, \mathbb{S}) \cong \widehat{\mathcal{F}}$.

To show that this linear isomorphism is furthermore a bimodule map, we need only confirm that the isomorphism (9.3) used in the proof of Theorem 9.5 is compatible with the saturation map (6.11), that is, the commutativity of the diagram

$$\begin{CD} \widehat{H}^L(\lambda \mathbb{X}_{\lambda'}^L) @>(6.11)>> \widehat{H}^G(\lambda \mathbb{X}_{\lambda'}) \\ @V(9.3)VV @VV(9.3)V \\ e^{\mathbb{Z}}(\nu') \cdot \widehat{\text{fT}}^\varphi \cdot e^{\mathbb{Z}}(\nu) @>>> e(\gamma') \cdot \widehat{\text{fT}}^\varphi \cdot e(\gamma), \end{CD}$$

where the lower horizontal arrow is simply forgetting the \mathbb{Z} -component of the longitude, and λ, λ' are lifts of ν, ν' to $\mathfrak{t}_L + \varphi \cong \mathfrak{t} + \varphi$.

In order to see this, we need to check that the images of the minimal diagrams, single crossings and single dots are related by the map (6.11). This is clear from the definition of (9.3), which ultimately depends on the map given in [Web19b, Theorem 4.5]. The inverse of (9.3) sends each dot to the Chern class of a tautological bundle, and each single crossing to the saturation of a homology class in $\zeta \mathbb{X}_{\zeta'}^{L'}$ for a Levi $L' \subset L$ with one simple root and $\zeta \in \mathfrak{t}_{\mathbb{Z}}/W_L$ arbitrary. By the transitivity of saturation, this crossing gives classes in $\zeta \mathbb{X}_{\zeta'}^L$ and $\zeta \mathbb{X}_{\zeta}$ which are compatible under saturation. \square

From the discussion above, we can restate Theorem 5.10 using the functors of Definition 8.9 as:

Theorem 9.12. *The equivalence of Theorem 9.5 intertwines the functors ind_{ξ} and res_{ξ} with the functors of induction and restriction for flavoured KLRW algebras:*

$$\begin{CD} \widetilde{\text{fT}}_{\mathcal{S}}^\varphi\text{-wgm} @>\text{Res}>> \bigotimes_{p \in \mathbb{Z}} \widetilde{\text{fT}}_{\mathcal{S}^{(p)}}^\varphi\text{-wgm} \\ @V \mathcal{W} VV @VV \mathcal{W}^L V \\ \mathcal{A}(G, N) - \Gamma \Pi_{\mathcal{S}} @>\text{res}_{\xi}>> \mathcal{A}(L, N_0^{\xi}) - \Gamma \Pi_{\mathcal{S}^L} \end{CD}$$

9.3. Category \mathcal{O}

Now, we consider how these results apply to category \mathcal{O} , as described in Section 4.3.

Let us temporarily return to the general context of Section 5, with our usual notation G, N, ξ, L . We also choose a character $\chi: G \rightarrow \mathbb{C}^\times$, which defines a category $\mathcal{A}(G, N)\text{-}\mathcal{O}$. The standard choice for a quiver gauge theory, following Nakajima, is given by the product of the determinant characters on $GL(v_i)$.

The character χ can be restricted to L , and thus also defines a category $\mathcal{A}(L, N_0^{\xi})\text{-}\mathcal{O}$ of modules over $\mathcal{A}(L, N_0^{\xi})$. Unfortunately, the interaction of category \mathcal{O} and restriction functors is quite complicated.

Lemma 9.13. Consider a module $M \in \mathcal{A}(G, N)\text{-}\mathcal{O}$.

1. If $\chi(\xi) \geq 0$, then $\text{res}_\xi(M) = 0$.
2. If $\chi(\xi) < 0$, then $\text{res}_\xi(M) \notin \mathcal{A}(L, N_0^\xi)\text{-}\mathcal{O}$ unless $\text{res}_\xi(M) = 0$. However,

$$\text{Supp res}_\xi(M) = \text{Supp}(M) + \mathbb{Z}\xi.$$

Here, we use the support of a Gelfand-Tsetlin module from Definition 4.1. Of course, the same result holds if we reverse all signs.

Proof. Fix any weight $\nu \in \tilde{\mathfrak{t}}/W_L$. For all sufficiently large $k \in \mathbb{Z}_{\geq 0}$, we have

$$\mathcal{W}_\nu(\text{res}_\xi(M)) \cong \mathcal{W}_{\nu-k\xi}(\text{res}_\xi(M)) \cong \mathcal{W}_{\nu-k\xi}(M),$$

where the first isomorphism comes from the fact that $r_\xi^k : \mathcal{W}_{\nu-k\xi} \cong \mathcal{W}_\nu$ is an isomorphism of weight functors for $\mathcal{A}(L, N_0^\xi)$ and the second isomorphism comes from the fact that $\nu - k\xi$ is ξ -negative, and so Theorem 5.8 applies.

If $\chi(\xi) > 0$, then for sufficiently large k , $\mathcal{W}_{\nu-k\xi}(M) = 0$ (since M is in category \mathcal{O}) and thus $\mathcal{W}_\nu(\text{res}_\xi(M)) = 0$. Since ν is arbitrary, this forces $\text{res}_\xi(M) = 0$.

Assume now that $\chi(\xi) = 0$. Since M is in category \mathcal{O} , M^P (see Section 5.1) is in category \mathcal{O} for $\mathcal{A}(L, N)$. Then consider the system (5.1)

$$\dots \xrightarrow{r_\xi} \mathcal{W}_{\nu+\xi}^{L,N}(M^P) \xrightarrow{r_\xi} \mathcal{W}_\nu^{L,N}(M^P) \xrightarrow{r_\xi} \mathcal{W}_{\nu-\xi}^{L,N}(M^P) \xrightarrow{r_\xi} \dots$$

Since $\chi(\xi) = 0$, each space in this system lies in the same χ -eigenspace. Since M^P is in category \mathcal{O} , the χ -eigenspaces are finite-dimensional. Thus, only finitely many spaces in this system are nonzero. Hence, by Lemma 5.5, we conclude that $\mathcal{W}_\nu(\text{res}_\xi(M)) = 0$.

If $\chi(\xi) < 0$, then the above observation implies that $\text{Supp}(\text{res}_\xi(M)) = \text{Supp}(M) + \mathbb{Z}\xi$. In particular, this means that the eigenspaces of ξ cannot be bounded below. □

Now, we return to the quiver case, where we can make some more precise statements. Recall that $\tilde{\text{fl}}_\mathcal{S}^\varphi$ has a steadied quotient $\text{fl}_\mathcal{S}^\varphi$, defined in Definition 7.17. Applying Proposition 4.10 in the quiver case, we find that:

Theorem 9.14. Fix a flavour φ and an orbit \mathcal{S} . A module $M \in \mathcal{A}_\varphi(\mathbf{v}, \mathbf{w})\text{-}\Gamma\Pi_\mathcal{S}$ lies in category \mathcal{O} if and only if the corresponding $\tilde{\text{fl}}_\mathcal{S}^\varphi$ module factors through the steadied quotient $\text{fl}_\mathcal{S}^\varphi$. Thus, $\mathcal{A}_\varphi(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_\mathcal{S} \cong \text{fl}_\mathcal{S}^\varphi\text{-mod}$.

Proof. The proof is the same as that of [KTW⁺19b, Theorem 5.21]. □

Let H be a real number. We say that $\gamma \in \mathfrak{t}/W$ is **H -bounded above** if for all $i \in I$ and $z \in \gamma_i$, we have $\Re(z) \leq H$. The following result can be viewed as a specialization of Proposition 4.9 to the quiver situation:

Lemma 9.15. Fix a flavour φ and an orbit \mathcal{S} . For H sufficiently large, each $M \in \mathcal{A}_\varphi(\mathbf{v}, \mathbf{w})\text{-}\Gamma\Pi_\mathcal{S}$ satisfies:

$$M \in \mathcal{A}_\varphi(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_\mathcal{S} \quad \text{if and only if} \quad \text{every } \gamma \in \text{Supp } M \text{ is } H\text{-bounded above.}$$

Proof. Assume that every $\gamma \in \text{Supp } M$ is H -bounded above. Let $n = \sum v_i$. Let $C \subset \mathcal{S}$ be an equivalence class, such that $\mathcal{W}_\gamma(M) \neq 0$ for some $\gamma \in C$. Since all elements of C are H -bounded above, the function $\gamma \mapsto \Re(\chi(\gamma))$ is bounded above on \mathcal{S} by $H(n - 1)$. Moreover, in C , there are only finitely many γ , such that $\Re(\chi(\gamma))$ is larger than any given real number. Thus, C is $-$ bounded in the sense of Section 4.3 and so, by Proposition 4.9, we have $M \in \mathcal{A}_\varphi(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_\mathcal{S}$.

Now assume that M is in $\mathcal{A}_\varphi(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_\mathcal{S}$. Let $\gamma \in \text{Supp } M$. Let $H' = \max(|\Re(\varphi_e)|)$. We claim that γ is $H = 2H'n$ -bounded above.

Let $(\mathbf{i}, \mathbf{a}, \langle \rangle)$ be a flavoured sequence corresponding to γ by Lemma 7.9. Suppose that $\mathfrak{R}(a_k) > 2H'n$ (if not, we are done). Then the intervals $[a_k - H', a_k + H']$ (for $k = 1, \dots, n$) cannot cover $[0, 2H'n]$, so there must be a real number $H'' \in [0, 2H'n]$, such that there is no $\mathfrak{R}(a_k)$ in the interval $[H'' - H', H'' + H']$. This means that all the strands with real longitude $> H'' + H'$ have all ghosts with real longitude $> H''$, and all strands with real longitude $< H'' - H'$ have all ghosts with real longitude $< H''$.

Then consider $\{g \in \text{CGR} : \mathfrak{R}(a_g) > H''\}$. This set is nonempty and consists exactly of the last k elements of CGR for some k and all their ghosts. Thus, we see that $(\mathbf{i}, \mathbf{a}, \langle \rangle)$ is unsteady, and hence, $\mathcal{W}_\gamma(M) = 0$ by Theorem 9.14. □

Recall the notation ξ and associated $\mathbf{v}^{(p)}$ as defined in Section 3.2 and used in the previous section. From Lemma 9.13, it is possible to show that if $\mathbf{v}^{(p)} \neq 0$ for some $p > 0$, then $\text{res}_\xi(M) = 0$ for all M in category \mathcal{O} . Thus, we only consider those ξ for which $\mathbf{v}^{(p)} = 0$ for $p > 0$. In fact, we will focus on ξ , such that only $\mathbf{v}^{(0)}$ and $\mathbf{v}^{(-1)}$ are nonzero.

9.4. Divided power functors

Assume that Γ has no edge loops, and let us turn to rephrasing the divided power functors of Definition 8.13 in terms of Coulomb branches. Fix $i \in I$ and $k \leq v_i$. Consider a coweight $\xi: \mathbb{C}^\times \rightarrow \prod_j GL(v_j)$ which is the trivial extension of the coweight $t \mapsto \text{diag}(t, \dots, t, 1, \dots, 1) \in GL(v_i)$ with k diagonal entries equal to t and $v_i - k$ entries 1. For this choice of ξ , $\mathbf{v}^{(-1)}$ is k times in the unit vector on i , and $\mathbf{v}^{(0)} = \mathbf{v} - \mathbf{v}^{(-1)} = \mathbf{v} - ke_i$.

Restriction (Definition 5.1) defines a functor

$$\text{res}_\xi: \mathcal{A}(\mathbf{v}, \mathbf{w})\text{-mod} \rightarrow \mathcal{A}(\mathbf{v} - ke_i, \mathbf{w}) \otimes \mathcal{A}(GL_k, 0)\text{-mod} \tag{9.5}$$

By definition, $\mathcal{A}(GL_k, 0)$ is the equivariant homology of the affine Grassmannian of GL_k , which is isomorphic to the quantum Toda algebra by [BF08, Theorem 3].

9.4.1. Nonintegral decomposition

Let us specialize to the case $k = 1$ momentarily. The Hamiltonian reduction of $\mathcal{A}(\mathbb{C}^\times, 0) = D(\mathbb{C}^\times)$ by r_ξ is \mathbb{C} (equivalently, we can use Theorem 2.11), and so we get a Hamiltonian reduction functor

$$\mathcal{A}(\mathbf{v} - e_i, \mathbf{w}) \otimes \mathcal{A}(\mathbb{C}^\times, 0)\text{-mod} \rightarrow \mathcal{A}(\mathbf{v} - e_i, \mathbf{w})\text{-mod} \quad M \mapsto M/(r_\xi - 1)M$$

as discussed in Section 5.4.

We define

$$\text{res}_i: \mathcal{A}(\mathbf{v}, \mathbf{w})\text{-mod} \rightarrow \mathcal{A}(\mathbf{v} - e_i, \mathbf{w})\text{-mod}$$

to be the composition of (9.5) with this Hamiltonian reduction functor. By Lemma 5.3 and Proposition 5.11, the functor res_i takes Gelfand-Tsetlin modules to Gelfand-Tsetlin modules.

We apply the paragraph below Proposition 5.11 (with the natural choice of $a = \xi$ after identifying $\mathfrak{t} = \mathfrak{t}^*$ using the trace form), and so we can decompose the Hamiltonian reduction over $c \in \mathbb{C}^\times$. This gives us a decomposition of functors $\text{res}_i = \sum_{c \in \mathbb{C}^\times} \text{res}_{i,c}$. Moreover, we have $\text{res}_{i,c}(M) = \mathcal{W}_{\log c}^a(\text{res}_\xi(M))$, where $\mathcal{W}_{\log c}^a$ denotes the weight space for $\mathcal{A}(\mathbb{C}^\times, 0)$.

9.4.2. Integral case

Now, we return to general k and further examine the case of integral Gelfand-Tsetlin modules.

Let $\text{res}_i^{(k)}: \mathcal{A}(\mathbf{v}, \mathbf{w}) - \Gamma\Pi_{\mathbb{Z}} \rightarrow \mathcal{A}(\mathbf{v} - ke_i, \mathbf{w}) - \Gamma\Pi_{\mathbb{Z}}$ be the functor res_ξ of (9.5), followed by passing to the generalized weight space \mathcal{W}_0 for $\mathcal{A}(GL_k, 0)$ (if we specialize $k = 1$, then we see that $\text{res}_i^{(1)} = \text{res}_{i,1}$).

Lemma 9.16. *The functor $\text{res}_i^{(k)}$ restricts to a functor $\mathcal{A}(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}} \rightarrow \mathcal{A}(\mathbf{v} - ke_i, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}}$.*

Proof. Let $M \in \mathcal{A}(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}}$. By Lemma 9.15, there exists N , such that every γ in $\text{Supp}(M)$ is N -bounded above.

We claim that every γ in $\text{Supp}(\text{res}_i^{(k)}(M))$ is also N -bounded above. By definition

$$\mathcal{W}_{\gamma}(\text{res}_i^{(k)}(M)) = \mathcal{W}_{\gamma \cup \{0, \dots, 0\}}(\text{res}_{\xi}(M)).$$

Since $\text{Supp}(\text{res}_{\xi}(M)) = \text{Supp}(M) + \mathbb{Z}\xi$, we see that $\gamma \cup \{m, \dots, m\} \in \text{Supp}(M)$ for some $m \in \mathbb{Z}$. In particular, this means that $\gamma \cup \{m, \dots, m\}$ is N -bounded above, so for all j and $x \in \gamma_j$, we have $\mathfrak{R}(x) \leq N$. Thus, γ is also N -bounded above (here, $\gamma \cup \{m, \dots, m\}$ means that we add k copies of m to γ_i). □

The same proof shows that our functors $\text{res}_{i,c}$ from the previous section all preserve category \mathcal{O} as well, so this is not a special property of the integral case.

In order to compute the left adjoint of $\text{res}_i^{(k)} : \mathcal{A}(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}} \rightarrow \mathcal{A}(\mathbf{v} - ke_i, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}}$, it is helpful to think of it as a composition of three functors:

1. The inclusion from category \mathcal{O} into all GT modules; the left adjoint to this is the functor that takes the largest quotient of a GT module lying in category \mathcal{O} .
2. The functor res_{ξ} , with its usual left adjoint ind_{ξ} .
3. The functor of passing to a weight space for the action of the Gelfand-Tsetlin subalgebra in $\mathcal{A}(GL_k, 0)$. This does not have a left adjoint in the category of Gelfand-Tsetlin modules, but it does have one in the category of pro-Gelfand-Tsetlin modules: outer tensoring a $\mathcal{A}(\mathbf{v} - ke_i, \mathbf{w})$ -module with the projective P_0 representing the functor \mathcal{W}_0 .

While $\text{ind}_{\xi}(M \boxtimes P_0)$ is only a pro-Gelfand-Tsetlin module, its maximal quotient in category \mathcal{O} is honestly Gelfand-Tsetlin, and thus defines a left adjoint functor to $\text{res}_i^{(k)}$, which we denote $\text{ind}_i^{(k)}$.

9.4.3. Compatibility with flavoured KLRW algebra functors

Now, assume that φ is an integral flavour.

Theorem 9.17. *The equivalence of Theorem 9.5 intertwines the divided power functor $\mathcal{E}_i^{(k)}$ with the functor $\text{res}_i^{(k)}$ defined above, and thus its left adjoint $\mathcal{F}_i^{(k)}$ with $\text{res}_i^{(k)}$:*

$$\begin{array}{ccc}
 \mathfrak{fl}_{\mathbf{v}}^{\varphi}\text{-mod} & \xrightarrow{\mathcal{E}_i^{(k)}} & \mathfrak{fl}_{\mathbf{v}-ke_i}^{\varphi}\text{-mod} \\
 \uparrow \mathcal{W} & & \uparrow \mathcal{W} \\
 \mathcal{A}_{\varphi}(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}} & \xrightarrow{\text{res}_i^{(k)}} & \mathcal{A}_{\varphi}(\mathbf{v} - ne_i, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}}
 \end{array}$$

$$\begin{array}{ccc}
 \mathfrak{fl}_{\mathbf{v}}^{\varphi}\text{-mod} & \xleftarrow{\mathcal{F}_i^{(k)}} & \mathfrak{fl}_{\mathbf{v}-ke_i}^{\varphi}\text{-mod} \\
 \uparrow \mathcal{W} & & \uparrow \mathcal{W} \\
 \mathcal{A}_{\varphi}(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}} & \xleftarrow{\text{ind}_i^{(k)}} & \mathcal{A}_{\varphi}(\mathbf{v} - ne_i, \mathbf{w})\text{-}\mathcal{O}_{\mathbb{Z}}
 \end{array}$$

In particular, these functors define a categorical \mathfrak{g}_Γ -action which sends the weight $\mu = \sum w_i \varpi_i - v_i \alpha_i$ to the category $\mathcal{A}_\varphi(\mathbf{v}, \mathbf{w}) - \mathcal{O}_{\mathbb{Z}}$.

Proof. From Theorem 9.12, we have the commutative diagram

$$\begin{array}{ccc}
 \widetilde{\text{fl}}_{\mathbf{v}}^\varphi\text{-wmod} & \longrightarrow & \widetilde{\text{fl}}_{\mathbf{v}-ke_i}^\varphi \otimes \widetilde{\text{fl}}_{ke_i}^\varphi\text{-wmod} \\
 \uparrow & & \uparrow \\
 \mathcal{A}_\varphi(\mathbf{v}, \mathbf{w}) - \Gamma\Pi_{\mathbb{Z}} & \longrightarrow & (\mathcal{A}_\varphi(\mathbf{v} - ke_i, \mathbf{w}) \otimes \mathcal{A}_\varphi(ke_i, 0)) - \Gamma\Pi_{\mathbb{Z}}.
 \end{array} \tag{9.6}$$

From Section 8.3, we see that $\widetilde{\text{fl}}_{ke_i}^\varphi$ is the nilHecke algebra NH_k . Also, $\mathcal{E}_i^{(k)}$ is defined by following the upper horizontal arrow in (9.6) and then applying an idempotent in NH_k .

On the other hand, $\text{res}_i^{(k)}$ is defined by following the lower horizontal line in (9.6) and then taking \mathcal{W}_0 for $\mathcal{A}_\varphi(ke_i, 0) = \mathcal{A}(GL_k, 0)$. So then the result follows from Corollary 9.6. \square

Remark 9.18. Since the functor $\mathcal{F}_i^{(k)}$ is also isomorphic to the right adjoint of $\mathcal{E}_i^{(k)}$ (but not canonically so), we also have that $\text{ind}_i^{(k)}$ is isomorphic to the right adjoint of $\text{res}_i^{(k)}$, which we can construct as tensoring with the injective (infinitely generated) GT module P_0^* corepresenting \mathcal{W}_0 over $\mathcal{A}(GL_k, 0)$, applying $\text{coind}_{\mathcal{E}}$, and then passing to the largest submodule in \mathcal{O} .

Recall the idempotent $e_H \in \widetilde{\text{fl}}_{\mathbf{v}}^\varphi$ from Proposition 7.26. From Lemma 4.11, we have that:

Proposition 9.19. *Let $M \in \mathcal{A}(\mathbf{v}, \mathbf{w}) - \mathcal{O}_{\mathbb{Z}}$. We have that $e_H \mathcal{W}(M) = 0$ if and only if the Gelfand-Kirillov dimension of M is strictly less than $\frac{1}{2} \dim M_C(\mathbf{v}, \mathbf{w}) = \sum_{i \in I} v_i$.*

Proof. If $e_H \mathcal{W}(M) \neq 0$, then there exists \mathbf{i} , such that $e(\mathbf{i}, H) \mathcal{W}(M) \neq 0$. By the definition of $e(\mathbf{i}, H)$, this means that if we construct $\gamma^H \in \mathfrak{t}/W$, where $\gamma_i^H = \{kH : i_k = i\}$, then $\mathcal{W}_{\gamma^H}(M) \neq 0$. Now, choose any integers a_1, \dots, a_n , such that $a_k \leq a_{k+1} + H$ and then define γ , such that $\gamma_i = \{a_k : i_k = i\}$. We have an isomorphism of functors $\mathcal{W}_{\gamma^H}(M) \cong \mathcal{W}_\gamma(M)$ since the corresponding idempotents in the flavoured KLRW algebra are equivalent, and thus, $\mathcal{W}_\gamma(M) \neq 0$. As the set of such γ is Zariski dense in \mathfrak{t}/W , we see that M has Gelfand-Kirillov dimension $\sum_{i \in I} v_i$ by Lemma 4.11.

If $e_H \mathcal{W}(M) = 0$, then any γ in the support of M must contain a pair y, z , such that the difference $|y - z|$ is an integer with $|y - z| < H$. If there is no such pair, then $e'(\gamma)$ is isomorphic to an idempotent $e'(\gamma')$ satisfying $e'(\gamma') = e_H e'(\gamma')$. The locus where $|y - z| \in \{-H + 1, -H + 2, \dots, H - 1\}$ holds is the union of a finite number of hyperplanes. Thus, the support of M is not Zariski dense. Thus, $\text{GKdim}(M) < \sum_{i \in I} v_i$ by Lemma 4.11 again. \square

Recall that $\mathcal{O}_{\text{top}}(\mathbf{v}, \mathbf{w})$ is the quotient of $\mathcal{A}(\mathbf{v}, \mathbf{w}) - \mathcal{O}_{\mathbb{Z}}$ by the subcategory of objects with GK dimension $< \frac{1}{2} \dim M_C(\mathbf{v}, \mathbf{w})$. Combining Proposition 7.26, Corollary 9.6 and Proposition 9.19, we find that:

Corollary 9.20. *The functor $e_H \mathcal{W}$ induces an equivalence $\mathcal{O}_{\text{top}}(\mathbf{v}, \mathbf{w}) \cong R_{\mathbf{v}}^{\mathbf{w}}$ -mod, and thus an isomorphism of \mathfrak{g}_Γ -modules $\bigoplus_{\mathbf{v}} K_{\mathbb{C}}(\mathcal{O}_{\text{top}}(\mathbf{v}, \mathbf{w})) \cong V(\lambda)$.*

9.4.4. The nonintegral case

Let us return to setting nonintegral GT modules, but specialize $k = 1$, as in Section 9.4.1. There, we defined a decomposition $\text{res}_i = \sum_{c \in \mathbb{C}^\times} \text{res}_{i,c}$, which we will now examine from the flavoured KLRW perspective.

We have a functor $\text{fl}_n^\varphi\text{-mod} \rightarrow \text{fl}_{n-1}^\varphi \otimes \widetilde{\text{fl}}_1\text{-mod}$, where the subscript simply denotes the number of corporeal strands. The single strand in the second factor has a longitude z . Each nonzero summand that arises in the decomposition of this functor on a given block $\text{fl}_n^\varphi\text{-mod}$ is given by a vertex $(i, [z])$ of $\widetilde{\Gamma}$, where $\widetilde{\Gamma}$ is defined using \mathcal{S} in Definition 7.19. Thus, we obtain functors $\mathcal{E}_{i,[z]} : \text{fl}_n^\varphi\text{-mod} \rightarrow \text{fl}_{n-1}^\varphi\text{-mod}$, which match the above functors $\text{res}_{i,c}$ (where $c = \exp(2\pi iz)$), as in Theorem 9.17. This shows that if $\text{supp } M$ lies in an orbit \mathcal{S} and $\text{res}_{i,c}(M)$ is nonzero, then $(i, [\log c]) \in \widetilde{\Gamma}$.

Consider the graph $\Gamma \times \mathbb{C}/\mathbb{Z}$ with the adjacency of Definition 7.19. For simplicity, we assume that this graph has no edge loops, even if Γ has an edge loop. Consider the graph $\Gamma_\varphi \subset \Gamma \times \mathbb{C}/\mathbb{Z}$ is given by the union of $\tilde{\Gamma}$ for all orbits \mathcal{S} that appear in the support of modules in \mathcal{O} for our fixed φ and all different values of \mathbf{v} .

Using Remark 8.16 and repeating the proof of Corollary 9.20 shows:

Proposition 9.21. *The functors $\text{res}_{i,c}$ give a categorical action of $\mathfrak{g}_{\Gamma_\varphi}$ on $\bigoplus_{\mathbf{v}} \mathcal{A}(\mathbf{v}, \mathbf{w})\text{-}\mathcal{O}$. The corresponding Grothendieck groups $\bigoplus_{\mathbf{v}} K_{\mathbb{C}}(\mathcal{O}_{\text{top}}(\mathbf{v}, \mathbf{w}))$ form an irreducible representation of $\mathfrak{g}_{\Gamma_\varphi}$.*

Let U be the full subgraph of $\Gamma \times \mathbb{C}/\mathbb{Z}$ containing all those vertices connected to $(i, [\varphi])$ for φ the weight of an edge connecting the vertex i to the Crawley-Boevey vertex. Note that if φ is integral, then $U = \Gamma \times \{[0]\}$.

Lemma 9.22. *We have an equality $U = \Gamma_\varphi$ and the functor $\text{res}_{i,c}$ is not zero if and only if $(i, [\log c]) \in \Gamma_\varphi$.*

This implies that if we take $M \in \mathcal{A}(\mathbf{v}, \mathbf{w})\text{-}\Gamma\Pi_{\mathbb{Z}}$ an integral Gelfand-Tsetlin module, then $\text{res}_i(M) = \text{res}_{i,1}(M)$, as $\text{res}_{i,c}(M) = 0$ for $c \neq 1$.

Proof. To show the inclusion $U \subset \Gamma_\varphi$, it's enough to show that any $(i, [\log c]) \in U$ occurs as a longitude in a nonzero idempotent in fl_n^φ for some n . As before, we consider a highest weight λ , such that $\alpha_{j,[z]}^\vee(\lambda) = \tilde{w}_{j,[z]}$. By Corollary 7.27, it's enough to show that the irreducible representation $V(\lambda)$ has a nontrivial weight space for a weight μ with $\alpha_{i,[\log c]}^\vee(\mu) < 0$. Since $\alpha_{i,[\log c]}$ is conjugate to all other simple roots in its component under the action of the Weyl group, we can choose w in the Weyl group of Γ_φ , such that $-w\alpha_{i,[\log c]} = \alpha_{k,[y]}$ is a simple root with $\alpha_{k,[y]}^\vee(\lambda) > 0$. This implies $\mu = w\lambda$ is a nonzero weight as desired, so the equality $\lambda = \mu + \sum_{(j,[z]) \in \Gamma_\varphi} \tilde{v}_{j,[z]} \alpha_{j,[z]}$ gives the desired dimension vector.

Now we will show that if $(i, [\log c]) \in U$, then $\text{res}_{i,c} \neq 0$. Note that our argument also shows that any vector of weight μ in $V(\lambda)$ has nonzero image under the action of the Chevalley generator $E_{i,[\log c]}$, since it has negative weight for the corresponding root $\mathfrak{s}l_2$. Thus, any module M with nonzero image in \mathcal{O}_{top} supported on the corresponding orbit \mathcal{S} satisfies $\text{res}_{i,c}(M) \neq 0$.

On the other hand, by Lemma 7.29, if $\text{fl}^\varphi(\mathcal{S}) \neq 0$ for some orbit \mathcal{S} , then every component of the corresponding graph $\tilde{\Gamma}$ has vertex $(j, [z])$ with nonzero $\tilde{w}_{j,[z]}$. Thus, $\tilde{\Gamma} \subset U$. Ranging over all \mathcal{S} , this shows that $\Gamma_\varphi \subset U$. We noted above that if $\text{res}_{i,c}(M)$ is nonzero, then $(i, [\log c]) \in \tilde{\Gamma}$. This completes the proof. \square

From a more topological perspective, the Crawley-Boevey graph $\Gamma_\varphi^{\tilde{\mathbf{w}}}$ can be built as follows:

1. Take the minimal cover of the Crawley-Boevey graph $\Gamma^{\mathbf{w}}$ that trivializes the flavour φ as an element of $H^1(\Gamma^{\mathbf{w}}, \mathbb{C}/\mathbb{Z})$.
2. Delete all but one preimage of the Crawley-Boevey vertex. This will be the Crawley-Boevey vertex of $\Gamma_\varphi^{\tilde{\mathbf{w}}}$.
3. Delete all components of the preimage of Γ not connected to the CB vertex.

9.5. Cherednik algebras

Throughout this section, we assume that Γ is the Jordan quiver, with dimension vector $v = n, w = \ell > 0$. As in the Introduction, we let $\mathcal{A}(n, \ell) = \mathcal{A}(G, N)$, where $G = GL(n)$ and $N = \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \text{Hom}(\mathbb{C}^n, \mathbb{C}^\ell)$. The case where $\ell = 0$ is somewhat different and requires slightly different methods.

Our aim in this subsection is to prove Theorem 1.5. In order to do this, let us unpack the results of the previous sections in this case. Our flavour consists of a ℓ -tuple $\varphi_1, \dots, \varphi_\ell$ corresponding to the torus of $GL(\mathbb{C}^\ell)$, and a single scalar $\mathbf{t} \in \mathbb{C}$, which corresponds to the \mathbb{C}^\times action scaling the loop in Γ . As discussed in Section 1.4, Kodera and Nakajima have defined an isomorphism of algebras $\text{KN}: \text{H}(n, \ell) \rightarrow \mathcal{A}(n, \ell)$. This isomorphism matches the usual Euler grading on $\text{H}(n, \ell)$ to the grading on $\mathcal{A}(n, \ell)$ induced by the isomorphism $\pi_1(GL_n) \cong \mathbb{Z}$, so it induces an equivalence $\text{KN}: \text{H}(n, \ell)\text{-}\mathcal{O} \rightarrow \mathcal{A}(n, \ell)\text{-}\mathcal{O}$. This

isomorphism relates the flavour parameters to the usual parameters of the Cherednik algebra by formulas given in [KN18, Theorem 1.1] (where φ_r is denoted z_r). We let $H_\varphi(n, \ell)$ be the spherical Cherednik algebra specialized at these parameters. Since we will want to compare with the results of [Web17b], we note that in that paper, the parameter \mathbf{t} is denoted by k , and the parameters φ_r by s_r .

Consider the graph with vertices \mathbb{C}/\mathbb{Z} with an edge from $[z] \rightarrow [z + \mathbf{t}]$, for all $[z] \in \mathbb{C}/\mathbb{Z}$. This is a union of e -cycles if $\mathbf{t} = m/e$ is rational, and a union of A_∞ -components if $\mathbf{t} \notin \mathbb{Q}$. For simplicity, we'll avoid the case where $\mathbf{t} \in \mathbb{Z}$, so this graph has no edge loops.

Given a \tilde{W} orbit $\mathcal{S} \subset \mathbb{C}^n$, we defined a graph $\tilde{\Gamma}$ in Definition 7.19. From the definition, we see that $\tilde{\Gamma}$ is always a subgraph of this \mathbb{C}/\mathbb{Z} -graph.

As discussed in Section 9.4.1 above, we have a decomposition of the induction and restriction functors $\text{ind}_{\bullet, c}, \text{res}_{\bullet, c}$ corresponding to the single vertex of the quiver (defined using the Hamiltonian reduction approach). On category \mathcal{O} , this gives a \mathfrak{g}_U action (by Proposition 9.21), where c ranges over the vertices of $U = \Gamma_\varphi$. The analogous decomposition of the Bezrukavnikov-Etingof functors into i -induction/restriction functors $\text{res}_{\text{BE}, i}$ is given by Shan [Sha1, Definition 4.2].

By Theorem 9.17, we have that $\mathcal{A}_\varphi(n, \ell)\text{-}\mathcal{O} \cong \mathfrak{fl}_n^\varphi\text{-mod}$, where n gives the number of corporeal strands. By Lemma 7.21, we have that \mathfrak{fl}_n^φ is isomorphic to a flavoured KLRW algebra over $\tilde{\Gamma}$, which we can think of a flavoured KLRW algebra for U by the inclusion of $\tilde{\Gamma}$ as a subgraph. Finally, we find that by Theorem 7.23, we have a quotient functor $\mathcal{W}_! : T_n^\theta\text{-mod} \rightarrow \mathcal{A}_\varphi(n, \ell)\text{-}\mathcal{O}$ from the category of modules over a weighted KLRW algebra T_n^θ for U , where the subscript n indicates that we have a total of n corporeal strands (possibly of different labels).

Let $H_\varphi(n, \ell)$ be the full Cherednik algebra for $G(\ell, 1, n)$. In [Web19a], the second-named author defined an equivalence of categories $\mathcal{W} : \mathfrak{H}_\varphi(n, \ell)\text{-}\mathcal{O} \rightarrow \tilde{T}_n^\theta\text{-mod}$ using a similar method to Theorem 9.5.

Lemma 9.23. *We have an isomorphism of functors $\mathcal{W}_!(-) \cong \text{KN}(eW^{-1}(-))$.*

Proof. To see this, we note that the three functors involved are uniquely characterized by matching three polynomial representations:

1. the polynomial representation of $\mathfrak{H}(n, \ell)$ defined in [Web19a, (2.17–2.22)], extending the representation of the spherical part given in [KN18, Theorem 1.5].
2. the GKLO representation of $\mathcal{A}(n, \ell)$.
3. the polynomial representation of \tilde{T}_n^θ , defined in [Web19b, Proposition 2.7].

The Kodera-Nakajima isomorphism can be defined as the unique one matching (1) and (2), the functor \mathcal{W} is uniquely defined by matching (2) and (3), and from the definition [Web19a, Lemma 3.12], we see that the functor \mathcal{W} is uniquely characterized by matching (1) and (3). This shows the desired compatibility. \square

However, at the moment, we do not know how to prove the compatibility of our induction and restriction functors with Bezrukavnikov and Etingof's under the equivalence of categories induced by the Kodera-Nakajima isomorphism. Instead, we have to use a potentially different equivalence of categories, which was explicitly constructed with this property in mind.

In [Web17b, Theorem 4.8], the second-named author constructed a potentially different equivalence $\mathfrak{B} : \mathfrak{H}_\varphi(n, \ell)\text{-}\mathcal{O} \rightarrow T_n^\theta\text{-mod}$ using the method of uniqueness of 1-faithful covers. Since this equivalence is constructed using [Web17b, Theorem 2.3], it is partly characterized by the fact that it intertwines the Bezrukavnikov-Etingof induction and restriction functors with the induction and restriction functors for weighted KLRW algebras induced by the inclusion $\tilde{T}_{n-1}^\theta \hookrightarrow \tilde{T}_n^\theta$. Note, this equivalence depends on a choice of isomorphism between the Hecke algebra of $G(\ell, 1, n)$ (which appears here as endomorphisms of the KZ functor) and the cyclotomic KLR algebra inside T_n^θ used in the application of [Web17b, Theorem 2.3], so it is not unique. We expect that if this isomorphism is chosen correctly, then it will induce an isomorphism of functors $\mathfrak{B} \cong \mathcal{W}$, but there are a number of variations on this isomorphism possible, and it is a difficult calculation to check if any of them is right.

Proof of Theorem 1.5. From the discussion above, we have functors:

$$\begin{array}{ccccc}
 \mathcal{A}_\varphi(n, \ell)\text{-}\mathcal{O} & \xleftarrow{\mathcal{W}_!} & \tilde{T}_n^\theta\text{-mod} & \xleftarrow{\mathfrak{B}} & \mathfrak{H}_\varphi(n, \ell)\text{-}\mathcal{O} \\
 \text{res}_\bullet \downarrow & & \text{Res} \downarrow & & \text{res}_{\text{BE}} \downarrow \\
 \mathcal{A}_\varphi(n-1, \ell)\text{-}\mathcal{O} & \xleftarrow{\mathcal{W}_!} & \tilde{T}_{n-1}^\theta\text{-mod} & \xleftarrow{\mathfrak{B}} & \mathfrak{H}_\varphi(n-1, \ell)\text{-}\mathcal{O}
 \end{array}$$

with both squares commuting. Thus, to complete the proof, we need only show that the induced quotient functor $\mathcal{W}_! \circ \mathfrak{B} : \mathfrak{H}_\varphi(n, \ell)\text{-}\mathcal{O} \rightarrow \mathcal{A}_\varphi(n, \ell)\text{-}\mathcal{O}$ kills precisely the aspherical modules, that is, those killed by the symmetrizing idempotent e .

We have a natural labelling of simple modules by multipartitions:

- in the category $\mathfrak{H}_\varphi(n, \ell)\text{-}\mathcal{O}$, we have standard modules $\Delta(\mu)$ induced from simple representations of $G(\ell, 1, n)$, and every simple $L(\mu)$ is a quotient of a unique such module.
- in the category $\tilde{T}_n^\theta\text{-mod}$, this labelling is induced by the cellular structure of [Web17b, Theorem 4.11].

The equivalence \mathfrak{B} intertwines these labellings by construction. On the other hand, we calculate directly in [Web19a, Lemma 3.17] how the Dunkl-Opdam subalgebra acts on the highest weight space of $L(\mu)$; this shows that \mathcal{W} preserves these labellings as well and hence $\mathfrak{B}(L(\mu)) \cong W(L(\mu))$ for all μ .

Thus, by Lemma 9.23, we have that

$$\mathcal{W}_!(\mathfrak{B}(L(\mu))) \cong \text{KN}(eL(\mu)),$$

so we have that $\mathcal{W}_!(\mathfrak{B}(L(\mu))) = 0$ if and only if $eL(\mu) = 0$. This shows that $\mathcal{W}_! \circ \mathfrak{B}$ factors through the usual quotient functor to $\mathfrak{H}_\varphi(n, \ell)\text{-}\mathcal{O}$, and induces an equivalence of that category to $\mathcal{A}_\varphi(n, \ell)\text{-}\mathcal{O}$ intertwining the restriction and induction functors, completing the proof. \square

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