J. Aust. Math. Soc. **108** (2020), 120–144
doi:10.1017/S1446788718000381

CONSERVED QUANTITIES ON MULTISYMPLECTIC MANIFOLDS

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(Received 1 September 2017; accepted 25 August 2018; first published online 26 December 2018)

Communicated by M. Murray

Abstract

Given a vector field on a manifold M, we define a *globally conserved quantity* to be a differential form whose Lie derivative is exact. Integrals of conserved quantities over suitable submanifolds are constant under time evolution, the Kelvin circulation theorem being a well-known special case. More generally, conserved quantities are well behaved under transgression to spaces of maps into M. We focus on the case of multisymplectic manifolds and Hamiltonian vector fields. Our main result is that in the presence of a Lie group of symmetries admitting a homotopy co-momentum map, one obtains a whole family of globally conserved quantities. This extends a classical result in symplectic geometry. We carry this out in a general setting, considering several variants of the notion of globally conserved quantity.

2010 *Mathematics subject classification*: primary 37K05, 53D05; secondary 70H33.

Keywords and phrases: conservation laws, multisymplectic manifolds, homotopy co-momentum map.

Introduction

The mathematical formulation and application of conserved quantities is most transparent in the case of classical point mechanics in its symplectic (or Hamiltonian) presentation. Given a symplectic manifold (M, ω) and a Hamiltonian function H in $C^\infty(M, \mathbb{R}) = \Omega^0(M)$, a function f on M is a 'conserved quantity' if the Lie derivative $\mathcal{L}_{v_H}(f) = -\{H, f\}$ vanishes, where v_H is the Hamiltonian vector field associated to H (that is, fulfilling $\iota_{v_H}\omega = -dH$) and $\{\,,\}$ is the Poisson bracket of (M, ω). If df is different from zero, the dimension of the phase space (M, ω) can be reduced by two and the associated Hamilton equation descends to the reduction. Iterating this process leads—at least locally—to the essentially trivial problem of solving a Hamilton equation on the real plane with its standard symplectic form. A typical source of conserved quantities is given by the Noether mechanism, here very simple: if a finite-dimensional Lie algebra $\mathfrak{g}$ acts on (M, ω) with a (co-)momentum map and the Hamiltonian function is $\mathfrak{g}$-invariant, then the image of every element of $\mathfrak{g}$ under the co-momentum is a conserved quantity.

The advent of a mathematically rigorous framework for observables and symmetries on a multisymplectic manifold (M, ω)—that is, a manifold with a closed, nondegenerate $(n+1)$-form for $n \geq 1$ (cf. [8, 9])—raises the question whether the above generalises from symplectic to multisymplectic geometry. Accordingly, we consider the set-up of a multisymplectic manifold (M, ω) and a Hamiltonian form $H \in \Omega^{n-1}(M)$, allowing for a vector field v_H such that $\iota_{v_H}\omega = -dH$. In fact, on a multisymplectic manifold, there are more general 'Hamilton(–de Donder–Weyl) equations': given a k-form η with $0 \leq k \leq n-1$, one asks for the existence of a multivector field v_η in $\Gamma_{C^\infty}(M, \Lambda^{n-k}TM)$ such that $\iota_{v_\eta}\omega = -d\eta$. The case $k = 0$ corresponds to classical field theories with n-dimensional sources (compare, for example, [6, Section 3.1] for these 'Hamiltonian n-curves').

Going back to the case that $\eta = H$ is a Hamiltonian $(n-1)$-form, we call a differential form $\alpha \in \Omega^\bullet(M)$ 'strictly conserved by H (or under v_H)' if $\mathcal{L}_{v_H}\alpha = 0$. Working with forms rather than functions, we immediately have two natural weakened notions: 'global conservation' (respectively 'local conservation') in case $\mathcal{L}_{v_H}\alpha$ is exact (respectively closed). Since conserved quantities are typically considered in integrated form, it is often enough that a 'quantity is preserved up to a total divergence', which corresponds to these two weakened notions, that are less interesting in the symplectic case. Among these three kinds of conserved quantities, the one we consider most useful are the globally conserved quantities.

As recalled earlier, in symplectic geometry all the components of co-momentum maps are conserved quantities. Our main goal is to understand to which extent a homotopy co-momentum associated to a multisymplectic action of a finite-dimensional Lie algebra $\mathfrak{g}$ (cf. [2, 10]) furnishes conserved quantities if $\mathfrak{g}$ keeps the Hamiltonian form H invariant. Given the more involved algebraic structure of the observables (and of the homotopy co-momentum), we find the following as the 'correct' generalisation of the above conservation law on symplectic manifolds (see Theorem 2.19).

THEOREM. *Let (M, ω) be a multisymplectic manifold, $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$, and (f) a homotopy co-momentum map for an infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M)$ which leaves H invariant. Let p be a k-cycle in the complex defining the Lie algebra homology of $\mathfrak{g}$. Then $f_k(p)$ is a globally conserved quantity.*

In fact, denoting the cycles of Lie algebra homology by $Z_k(\mathfrak{g})$ and the boundaries by $B_k(\mathfrak{g})$, we have the following table, whose last column reflects the above theorem (see Definition 2.5 for the three notions of preservation of H by the action of the Lie algebra $\mathfrak{g}$).

	H locally $\mathfrak{g}$-preserved	H globally $\mathfrak{g}$-preserved	H strictly $\mathfrak{g}$-preserved
$f_k(Z_k(\mathfrak{g}))$	locally conserved	locally conserved	globally conserved
$f_k(B_k(\mathfrak{g}))$	globally conserved	globally conserved	globally conserved

https://doi.org/10.1017/S1446788718000381 Published online by Cambridge University Press

We underline that all implications in the table are sharp, as we show by explicit examples. Further, our results hold even on relaxing the assumption that ω be multisymplectic, allowing ω to be any closed $(n+1)$-form.

In discussing conserved quantities (in the flavors strong/global/local), it turns out to be useful to first work in the more general situation of a manifold M together with a vector field v and to discuss differential forms 'preserved by this continuous dynamical system'. In this general context, we associate 'integral invariants' to conserved quantities by integrating the conserved forms over manifolds that are smoothly mapped to M. We obtain a very general form of Kelvin's classical circulation theorem that should be of use in continuum mechanics beyond the case of isentropic, incompressible fluids. More precisely, we have (compare with Theorem 4.1): *let Σ be a compact, oriented d-dimensional manifold (without boundary), v a vector field on M with flow ϕ_t, and $\sigma_0 : \Sigma \to M$ a smooth map. Consider $\sigma_t := \phi_t \circ \sigma_0 : \Sigma \to M$. If $\alpha \in \Omega^d(M)$ is a differential form, then the number*

$$\int_\Sigma (\sigma_t)^* \alpha$$

is independent of the time parameter t if α is globally conserved by v.

The above theorem makes apparent, in a geometric way, the usefulness of conserved quantities.

Let us now describe the content of the different sections in more detail. In Section 1 we introduce multisymplectic manifolds and define the various notions of conserved quantities. The heart of this note is Section 2: given a Lie group or Lie algebra action on a multisymplectic manifold that preserves (in one of the ways we make precise) a Hamiltonian form, we show that certain components of the homotopy co-momentum map are conserved quantities. Further, in Section 2.5 we provide an alternative, homological approach to prove these statements. In Section 3 we explain how the set-up needed in the previous section arises naturally and we make remarks on conserved quantities. We provide in Section 3.4 an example exhibiting a globally conserved quantity (with respect to a strictly H-preserving action) that is not strictly conserved. Finally, Section 4 is devoted to applications: we present a geometric version of Kelvin's circulation theorem and, more generally, we show that conserved quantities on a manifold M induce conserved quantities on spaces of maps into M.

A source of multisymplectic forms is field theory, but those considered there are of a very special kind, allowing notably for Darboux-type coordinates. For a discussion of the relation between conserved quantities in multisymplectic geometry on one side and classical field theory on the other side, we refer to Schreiber [11, Section 1.2.11].

1. Conserved quantities in multisymplectic geometry

The purpose of this section is to address conserved quantities associated to a vector field on a manifold. We will be mainly interested in the case that the manifold carries a multisymplectic structure and the vector field is Hamiltonian. We introduce these notions in Sections 1.1 and 1.2 and display some algebraic properties of conserved quantities in Section 1.3.

1.1. Multisymplectic manifolds.

DEFINITION 1.1. A manifold M equipped with a closed $(n+1)$-form $\omega \in \Omega^{n+1}(M)$ is called a *pre-n-plectic manifold*. It is called an *n-plectic* or *multisymplectic manifold* if the following map is injective for all $p \in M$:

$$T_pM \to \Lambda^n T_p^*M, \quad v \mapsto \iota_v \omega_p.$$

DEFINITION 1.2. Let (M, ω) be a pre-n-plectic manifold. An $(n-1)$-form α is called *Hamiltonian* if there exists a vector field $v_\alpha \in \mathfrak{X}(M)$ such that

$$d\alpha = -\iota_{v_\alpha}\omega.$$

We say that v_α is a *Hamiltonian vector field* for α. In the n-plectic case, v_α is unique. The set of Hamiltonian $(n-1)$-forms is denoted as $\Omega^{n-1}_{\mathrm{Ham}}(M)$.

REMARK 1.3. Observe that if v_α is a Hamiltonian vector field corresponding to α, then $\mathcal{L}_{v_\alpha}\omega = 0$ by Cartan's formula.

The following L_∞-algebra was constructed for n-plectic manifolds in [8, Theorem 5.2] and generalised to the pre-n-plectic case in [12, Theorem 6.7]. In the symplectic case, it reduces to the well-known Poisson algebra of functions.

DEFINITION 1.4. Given a pre-n-plectic manifold (M, ω), the *Lie n-algebra of observables* $L_\infty(M, \omega) = (L, \{l_k\})$ is the graded vector space given by

$$L_i = \begin{cases} \Omega^{n-1}_{\mathrm{Ham}}(M), & i = 0, \\ \Omega^{n-1-i}(M), & 0 < i \leq n-1, \end{cases}$$

together with the maps $\{l_k : L^{\otimes k} \to L \mid 1 \leq k \leq n+1\}$ given by

$$l_1(\alpha) = d\alpha \quad \text{if } \deg \alpha > 0,$$

$l_1(\alpha) = 0$ for $\deg \alpha = 0$, and, for all $k > 1$,

$$l_k(\alpha_1, \ldots, \alpha_k) = \begin{cases} 0 & \text{if } \deg \alpha_1 \otimes \cdots \otimes \alpha_k > 0, \\ \varsigma(k)\iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k})\omega & \text{if } \deg \alpha_1 \otimes \cdots \otimes \alpha_k = 0, \end{cases}$$

where v_{α_i} is a Hamiltonian vector field associated to $\alpha_i \in \Omega^{n-1}_{\mathrm{Ham}}(M)$ and $\varsigma(k) = -(-1)^{k(k+1)/2}$. Here the contraction with multivector fields is defined by $\iota(v_{\alpha_1} \wedge \cdots \wedge v_{\alpha_k}) = \iota_{v_{\alpha_k}} \ldots \iota_{v_{\alpha_1}}$.

1.2. Conserved quantities.

DEFINITION 1.5. Let M be a manifold and v a vector field on M. A form $\alpha \in \Omega^\bullet(M)$ is called a:

(a) *locally conserved quantity* if $\mathcal{L}_v\alpha$ is a closed form;
(b) *globally conserved quantity* if $\mathcal{L}_v\alpha$ is an exact form;
(c) *strictly conserved quantity* if $\mathcal{L}_v\alpha = 0$.

We denote the graded vector spaces of those quantities by $C_{\mathrm{loc}}(v)$, $C(v)$, and $C_{\mathrm{str}}(v)$, respectively.

Remark 1.6. In the sequel we will observe that condition (c) is very restrictive, whereas condition (a) is often too weak.

The following inclusions follow directly from Cartan's formula.

Lemma 1.7. *Let M be a manifold and v a vector field on M. Then:*

(i) $C_{\mathrm{str}}(v) \subset C(v) \subset C_{\mathrm{loc}}(v)$*;*

(ii) $\Omega^{\bullet}_{cl}(M) \subset C(v)$*;*

(iii) $d(C_{\mathrm{loc}}(v)) \subset C_{\mathrm{str}}(v)$*.*

We will be especially interested in the case where (M, ω) is pre-n-plectic and v preserves ω. In this case, additional results hold.

Lemma 1.8. *Let (M, ω) be a pre-n-plectic manifold and v a vector field on M such that $\mathcal{L}_v\omega = 0$. Then:*

(i) *$\alpha \in \Omega^{n-1}_{\mathrm{Ham}}(M)$ is locally conserved by v if and only if $\iota_{[v_\alpha, v]}\omega = 0$ for some (or equivalently for every) Hamiltonian vector field v_α for α.*

If moreover $v = v_H$ is a Hamiltonian vector field for $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$, then:

(ii) *$\alpha \in \Omega^{n-1}_{\mathrm{Ham}}(M)$ is locally conserved by v_H if and only if $\mathcal{L}_{v_\alpha}H$ is closed for some (or equivalently for every) Hamiltonian vector field v_α for α;*

(iii) *$\alpha \in \Omega^{n-1}_{\mathrm{Ham}}(M)$ is globally conserved by v_H if and only if $\mathcal{L}_{v_\alpha}H$ is exact for some (or equivalently for every) Hamiltonian vector field v_α for α;*

(iv) $H \in C(v_H)$*.*

Proof. Assertion (i) follows from the identity $\mathcal{L}_X \circ \iota_Y = \iota_Y \circ \mathcal{L}_X + \iota_{[X,Y]}$ applied to ω. Assertions (ii)–(iv) follow from Cartan's formula. □

Remark 1.9. We observe that the closedness (respectively exactness) of $\mathcal{L}_{v_\alpha}H$ is equivalent to the closedness (respectively exactness) of $l_2(\alpha, H)$.

As the following example illustrates, even in the n-plectic case, in general $\mathcal{L}_{v_H}H \neq 0$.

Example 1.10. Let $M = \mathbb{R}^3$, $\omega = dx \wedge dy \wedge dz$, and $H = x\,dy + z\,dz$. Then $v_H = -(\partial/\partial z)$, so $\iota_{v_H}H = -z$ and $\mathcal{L}_{v_H}H = -dz$.

Remark 1.11. In the symplectic (that is, the 1-plectic) case with $H \in C^\infty(M) = \Omega^0_{\mathrm{Ham}}(M)$, we have the following statements for $f \in C^\infty(M)$ and $v = v_H$:

(1) f is *globally conserved* if and only if f is *strictly conserved* and this is the case if and only if $\{H, f\} = 0$;

(2) f is *locally conserved* if and only if $\{H, f\}$ is locally constant.

As the following example shows, in the symplectic situation local conservedness does not suffice to formulate a 'conservation law'.

EXAMPLE 1.12. Let $M = \mathbb{R}^2$ with coordinates q, p, $\omega = dp \wedge dq$, and $H = p$. Taking $f = q$, the Hamiltonian vector field is given by $v_H = \partial/\partial q$ and thus $\mathcal{L}_{v_H} f = 1$, that is, f is locally but not globally conserved. Then, for any integral curve $\gamma(t) = (q_0 + (t - t_0), p_0)$ of v_H, we have $f(\gamma(t)) = f(\gamma(t_0)) + (t - t_0)$, that is, f is not a constant of motion.

1.3. The algebraic structure of conserved quantities. Throughout this subsection M will denote a manifold and v a vector field on M. We present here elementary methods to construct new conserved quantities from known ones.

LEMMA 1.13. *The space $C_{\mathrm{str}}(v)$ is a graded subalgebra of $\Omega^\bullet(M)$.*

As the following example illustrates, the spaces $C(v)$ and $C_{\mathrm{loc}}(v)$, unlike $C_{\mathrm{str}}(v)$, are not closed under wedge multiplication.

EXAMPLE 1.14. Let $M = \mathbb{R}^3$, $\omega = dx \wedge dy \wedge dz$, and $H = -x\,dy$. We observe that $dH = -dx \wedge dy$ and consequently $v_H = \partial/\partial z$. We set $\alpha = z\,dx$ and $\beta = z\,dy$. Then $\mathcal{L}_{v_H}\alpha = dx$ and $\mathcal{L}_{v_H}\beta = dy$ are exact but $\mathcal{L}_{v_H}(\alpha \wedge \beta) = 2z\,dx \wedge dy$ is not even closed.

However, stability under multiplication with elements from the following graded-commutative subalgebra of $\Omega^\bullet(M)$ is assured:

$$\mathcal{A}(v) := \{\beta \in \Omega(M) \mid d\beta = 0 \text{ and } \mathcal{L}_v\beta = 0\} \subset C_{\mathrm{str}}(v).$$

LEMMA 1.15. *The spaces $C(v)$ and $C_{\mathrm{loc}}(v)$ are graded modules over $\mathcal{A}(v)$.*

PROOF. We prove the statement for $C(v)$, the proof for $C_{\mathrm{loc}}(v)$ being identical. Let $\alpha \in C(v)$ (that is, there is a form γ with $\mathcal{L}_v\alpha = d\gamma$) and $\beta \in \mathcal{A}(v)$. Then

$$\mathcal{L}_v(\alpha \wedge \beta) = \mathcal{L}_v\alpha \wedge \beta + \alpha \wedge \mathcal{L}_v\beta = d\gamma \wedge \beta = d(\gamma \wedge \beta). \qquad \square$$

Again, more can be said if (M, ω) is pre-n-plectic and v preserves ω.

PROPOSITION 1.16. *Let (M, ω) be pre-n-plectic and v a vector field satisfying $\mathcal{L}_v\omega = 0$. The graded vector spaces*

$$L_\infty(M, \omega) \cap C_{\mathrm{loc}}(v), \quad L_\infty(M, \omega) \cap C(v), \quad \text{and} \quad L_\infty(M, \omega) \cap C_{\mathrm{str}}(v)$$

are L_∞-subalgebras of $L_\infty(M, \omega)$. Moreover, $\mathcal{L}_v(l_k(\beta_1, \ldots, \beta_k)) = 0$ for $k \geq 1$ and $\beta_1, \ldots, \beta_k \in L_\infty(M, \omega) \cap C_{\mathrm{loc}}(v)$.

PROOF. We claim that brackets of locally conserved quantities in $L_\infty(M, \omega)$ are strictly conserved. The only bracket which is nontrivial on components other than $\Omega^{n-1}_{Ham}(M)$ is $l_1 = d$. It follows from part (iii) of Lemma 1.7 that $l_1 = d$ applied to a locally conserved quantity is strictly conserved. Now for $k \geq 2$ consider $\beta_1, \ldots, \beta_k \in \Omega^{n-1}_{Ham}(M)$ such that $\mathcal{L}_v\beta_i$ is closed for all i. We want to show that

$$\mathcal{L}_v(l_k(\beta_1, \ldots, \beta_k)) = 0.$$

https://doi.org/10.1017/S1446788718000381 Published online by Cambridge University Press

As $l_k(\beta_1,\dots,\beta_k) = \pm\iota(v_{\beta_1}\wedge\cdots\wedge v_{\beta_k})\omega$, for any collection of Hamiltonian vector fields $\{v_{\beta_i}\}$ for $\{\beta_i\}$ this is equivalent to showing that

$$\mathcal{L}_v\iota_{v_{\beta_k}}\cdots\iota_{v_{\beta_1}}\omega = 0.$$

Using the identity $\mathcal{L}_X\circ\iota_Y = \iota_Y\circ\mathcal{L}_X + \iota_{[X,Y]}$, we can move $\mathcal{L}_v$ past the $\iota_{v_{\beta_i}}$ since $\iota_{[v,v_{\beta_i}]}\omega = 0$ by part (i) of Lemma 1.8. We find that

$$\mathcal{L}_v\iota_{v_{\beta_k}}\cdots\iota_{v_{\beta_1}}\omega = \iota_{v_{\beta_k}}\cdots\iota_{v_{\beta_1}}\mathcal{L}_v\omega = 0,$$

proving our claim. □

2. Conserved quantities from homotopy co-momentum maps

In this section we consider (infinitesimal) actions. More precisely, (M,ω) will always denote a pre-n-plectic manifold and G a Lie group (respectively $\mathfrak{g}$ a Lie algebra) acting on M.

Given a Hamiltonian form H on M, we define three notions of 'preservedness' of H with respect to the action; see Definition 2.5. Suitable conserved quantities are constructed from co-momentum maps for each of these three notions of preservedness, respectively, in Sections 2.2–2.4. Finally, in Section 2.5 we propose a less intuitive homological approach that has the advantage of being rather concise.

2.1. Actions on multisymplectic manifolds.

Definition 2.1. Let (M,ω) be a pre-n-plectic manifold. A right action ϑ of a Lie group G on M is called *multisymplectic* if $\vartheta_g^*\omega = \omega$ for all $g\in G$, where $\vartheta_g = \vartheta(\cdot,g)$. An infinitesimal right action of a Lie algebra $\mathfrak{g}$ on M, that is, a Lie algebra homomorphism $\mathfrak{g}\to\mathfrak{X}(M)$, $x\mapsto v_x$, is called *multisymplectic* if $\mathcal{L}_{v_x}\omega = 0$ for all $x\in\mathfrak{g}$. For a connected Lie group G, a right action ϑ is multisymplectic if and only if the corresponding infinitesimal right action (given by $x\mapsto v_x$, where $v_x(m) = d/dt|_0\vartheta(m,exp(tx))$ at all $m\in M$) is multisymplectic.

A multisymplectic infinitesimal action is thus a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{X}(M,\omega) = \{X\in\mathfrak{X}(M)|\mathcal{L}_X\omega = 0\}$. One may ask whether such an action admits an 'L_∞-lift' to $L_\infty(M,\omega)$. For an explicit description of the equations fulfilled by such a lift, the following definition is useful.

Definition 2.2. Let $\mathfrak{g}$ be a Lie algebra. We define the *Lie algebra homology differential* ∂ by setting

$$\partial_k = \partial|_{\Lambda^k\mathfrak{g}} : \Lambda^k\mathfrak{g}\to\Lambda^{k-1}\mathfrak{g},$$
$$x_1\wedge\cdots\wedge x_k \mapsto \sum_{1\le i<j\le k}(-1)^{i+j}[x_i,x_j]\wedge x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_k$$

for $k\geq 1$. We put $\Lambda^{-1}\mathfrak{g} = \{0\}$ and ∂_0 to be the zero map.

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We recall from [2, Section 5] the higher analogue of momentum map in symplectic geometry, obtained in a natural way by replacing Lie algebra morphisms by L_∞-algebra morphisms.

Definition 2.3. A *(homotopy) co-momentum map* for a multisymplectic infinitesimal action $v : \mathfrak{g} \to \mathfrak{X}(M)$ on (M, ω) is a collection of maps $(f) = \{f_i : \Lambda^i \mathfrak{g} \to \Omega^{n-i}(M) | 1 \leq i \leq n\}$ such that the generator of the action associated to $x \in \mathfrak{g}$ is a Hamiltonian vector field for $f_1(x)$ and satisfying the equation

$$-f_{k-1}(\partial(p)) = df_k(p) + \varsigma(k)\iota(v_p)\omega \tag{2.1}$$

for all $k = 1, \ldots, n+1$ and $p \in \Lambda^k \mathfrak{g}$ (setting f_0 and f_{n+1} to be zero). Here we use the shorthand notation $v_p := v_{x_1} \wedge \cdots \wedge v_{x_k}$ whenever $p = x_1 \wedge \cdots \wedge x_k$ for $x_i \in \mathfrak{g}$ and we set, as above, $\varsigma(k) = -(-1)^{k(k+1)/2}$.

Remark 2.4.

(1) Equation (2.1) of course is the general definition of an L_∞-algebra morphism specialised to the case at hand of the Lie algebra $\mathfrak{g}$ and the Lie n-algebra of observables $L_\infty(M, \omega)$.

(2) In [5, 10], (2.1) is interpreted as a co-boundary condition on a certain chain complex.

(3) A co-momentum map (f) is *G-equivariant* if the components $f_i : \Lambda^i \mathfrak{g} \to \Omega^{n-i}(M)$ are equivariant for all $i \in \{1, \ldots, n\}$. When G is connected, this can be expressed infinitesimally: for all $q \in \Lambda^i \mathfrak{g}$ and for all $x \in \mathfrak{g} = T_e G$, the equality $\mathcal{L}_{v_x}(f_i(q)) = f_i([x, q])$ holds. Here $[x, \cdot]$ is $ad(x)$ acting on $\Lambda^\bullet \mathfrak{g}$.

Now we turn to infinitesimal actions preserving a Hamiltonian $(n-1)$-form H on a pre-n-plectic manifold (M, ω). As in the case of the conserved quantities, one has to distinguish to which extent the action preserves the Hamiltonian form.

Definition 2.5. Let $\mathfrak{g} \to \mathfrak{X}(M, \omega), x \mapsto v_x$ be an infinitesimal action. It is called:

(a) *locally H-preserving* if $\mathcal{L}_{v_x} H$ is closed for all $x \in \mathfrak{g}$;

(b) *globally H-preserving* if $\mathcal{L}_{v_x} H$ is exact for all $x \in \mathfrak{g}$;

(c) *strictly H-preserving* if $\mathcal{L}_{v_x} H = 0$ for all $x \in \mathfrak{g}$.

Remark 2.6. Usually a differential form would be called 'preserved by an infinitesimal action' if condition (c) is fulfilled.

In the following, we will investigate the conserved quantities arising from co-momentum maps separately for these three cases.

2.2. Conserved quantities from locally H-preserving actions. In this subsection we assume that (M, ω) is a pre-n-plectic manifold, $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$, and that $(f) : \mathfrak{g} \to L_\infty(M, \omega)$ is the co-momentum of a *locally* H-preserving infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M, \omega), x \mapsto v_x$.

By the definition of a co-momentum map, the generator of the infinitesimal action associated to x in $\mathfrak{g}$ is a Hamiltonian vector field of $f_1(x)$. As earlier, for $p = x_1 \wedge \cdots \wedge x_k \in \Lambda^k \mathfrak{g}$, we write $v_p := v_{x_1} \wedge \cdots \wedge v_{x_k}$ and $\iota(v_p) = \iota_{v_{x_k}} \cdots \iota_{v_{x_1}}$.

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LEMMA 2.7. *Let* $(f) = \{f_i | 1 \le i \le n\}$ *be a co-momentum for* $v : \mathfrak{g} \to \mathfrak{X}(M, \omega)$ *and* $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$. *Then, for any Hamiltonian vector field* v_H *of* H*:*

(i) $f_1(x) \in C_{\mathrm{loc}}(v_H)$ *for all* $x \in \mathfrak{g}$;
(ii) $\iota_{[v_H, v_x]}\omega = 0$ *for all* $x \in \mathfrak{g}$;
(iii) $\iota(v_p)\omega \in C_{\mathrm{str}}(v_H)$ *for all* $p \in \Lambda^k\mathfrak{g}$.

PROOF. (i) Follows from Lemma 1.8(ii) and (ii) from Lemma 1.8(i). Further, (iii) follows upon recalling that $[\mathcal{L}_v, \iota_w] = \iota_{[v,w]}$ and (ii):

$$\mathcal{L}_{v_H}(\iota(v_p)\omega) = \mathcal{L}_{v_H}\iota_{v_{x_k}} \cdots \iota_{v_{x_1}}\omega = -\iota_{v_{x_k}}\mathcal{L}_{v_H} \cdots \iota_{v_{x_1}}\omega = \cdots = \pm\iota(v_p)(\mathcal{L}_{v_H}\omega) = 0. \qquad \square$$

It turns out that certain subspaces of the image of the higher components of the co-momentum map consist of locally conserved quantities. To specify this, we recall the definition of Lie algebra homology.

DEFINITION 2.8. Let $\mathfrak{g}$ be a Lie algebra, $k \ge 1$, and ∂_k the kth Lie algebra homology differential. We define:

(a) the *cycles* $Z_k(\mathfrak{g}) = \ker(\partial_k) \subset \Lambda^k\mathfrak{g}$;
(b) the *boundaries* $B_k(\mathfrak{g}) = \mathrm{im}(\partial_{k+1}) \subset \Lambda^k\mathfrak{g}$; and
(c) the kth *Lie algebra homology space* $H_k(\mathfrak{g}) = Z_k(\mathfrak{g})/B_k(\mathfrak{g})$.

REMARK 2.9. The space $Z_k(\mathfrak{g})$ is denoted by $\mathcal{P}_{\mathfrak{g},k}$ and called the *kth Lie kernel* of $\mathfrak{g}$ in [7].

PROPOSITION 2.10. *Let* $p \in Z_k(\mathfrak{g})$. *Then* $f_k(p)$ *is locally conserved by any Hamiltonian vector field* v_H *of* H.

PROOF. The case $k = 1$ is part (i) of Lemma 2.7. Assume now that $k > 1$. We have to show that $\mathcal{L}_{v_H} f_k(p)$ is closed. We have

$$d\mathcal{L}_{v_H} f_k(p) = \mathcal{L}_{v_H} d f_k(p) = -\varsigma(k)\mathcal{L}_{v_H}\iota(v_p)\omega = 0,$$

where the first equality holds because the Lie derivative commutes with the exterior derivative, the second one, because of (2.1), and the last one, because of Lemma 2.7(iii). $\square$

Proposition 2.10 states that $\mathcal{L}_{v_H} f_k(p)$ is a closed $(n-k)$-form; hence, we obtain the following corollary.

COROLLARY 2.11. *Let* $p \in Z_k(\mathfrak{g})$. *If* $H^{n-k}_{dR}(M)$ *is zero, then* $f_k(p)$ *is globally conserved by any Hamiltonian vector field* v_H *of* H.

For boundaries, the statement of Proposition 2.10 can be strengthened.

PROPOSITION 2.12. *If* $p \in B_k(\mathfrak{g}) \subset Z_k(\mathfrak{g})$, *then* $f_k(p)$ *is globally conserved by any Hamiltonian vector field* v_H *of* H.

https://doi.org/10.1017/S1446788718000381 Published online by Cambridge University Press

Proof. Let q be a potential for p, that is, $\partial_{k+1} q = p$. Then

$$\begin{aligned}\mathcal{L}_{v_H}(f_k(p)) = \mathcal{L}_{v_H}(f_k(\partial q)) &= \mathcal{L}_{v_H}(-d f_{k+1}(q) - \varsigma(k+1)\iota(v_q)\omega)\\ &= -d\mathcal{L}_{v_H} f_{k+1}(q) - \varsigma(k+1)\mathcal{L}_{v_H}\iota(v_q)\omega),\end{aligned}$$

using (2.1). The statement then follows by Lemma 2.7(iii). □

The following example shows sharpness of the statement of Proposition 2.10, that is, for $p \in \Lambda^k\mathfrak{g}$ the condition $\partial p = 0$, in general, does not imply that $f_k(p)$ is globally conserved.

Example 2.13. Let $M = \mathbb{R}^3$, $\omega = dx \wedge dy \wedge dz$, and $H = -x\,dy$. We already observed that $dH = -dx \wedge dy$ and $v_H = \partial/\partial z$. We consider the two-dimensional abelian Lie algebra $\mathfrak{g} = \langle a, b\rangle_{\mathbb{R}}$ and the homomorphism $v : \mathfrak{g} \to \mathfrak{X}(M)$ given by $v_a = \partial/\partial x$ and $v_b = \partial/\partial y$. We have that $\mathcal{L}_{v_a} H = -dy$ is exact and $\mathcal{L}_{v_b} H = 0$. We construct a co-momentum map for this action by $f_1(a) = -y\,dz$, $f_1(b) = x\,dz$, and $f_2(a \wedge b) = -z$. Then $a \wedge b \in Z_2(\mathfrak{g})$ is a cycle and $-z$ is locally conserved, as predicted by Proposition 2.10, but not globally:

$$\mathcal{L}_{v_H}(-z) = \iota_{\partial/\partial z} d(-z) = -1 \neq 0.$$

2.3. Conserved quantities from globally *H*-preserving actions. In this subsection we assume that (M, ω) is a pre-n-plectic manifold, $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$, and that $(f) : \mathfrak{g} \to L_\infty(M, \omega)$ is the co-momentum of a *globally* H-preserving infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M, \omega)$, $x \mapsto v_x$.

As Example 2.13 indicates, no significant improvements of the above results are to be expected upon passing from locally to globally H-preserving actions. There is only a slight improvement of Lemma 2.7(i) with essentially the same proof.

Lemma 2.14. *Let $(f) = \{f_i | 1 \leq i \leq n\}$ be a co-momentum for $v : \mathfrak{g} \to \mathfrak{X}(M)$. Then $f_1(x) \in C(v_H)$ for all $x \in \mathfrak{g}$ and for any Hamiltonian vector field v_H of H.*

Remark 2.15. Notice that Lemmas 2.7 and 2.14 hold for any element $\Omega^{n-1}_{\mathrm{Ham}}(M)$ whose Hamiltonian vector field is v_x.

2.4. Conserved quantities from strictly *H*-preserving actions. In this subsection we assume that (M, ω) is a pre-n-plectic manifold, $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$, and that $(f) : \mathfrak{g} \to L_\infty(M, \omega)$ is the co-momentum of a *strictly* H-preserving infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M, \omega)$, $x \mapsto v_x$. The assumption that the action is strictly H-preserving can be easily realised if there is a connected compact Lie group G with a smooth action on M whose differential is the infinitesimal action of $\mathfrak{g}$ on M; see Lemma 3.1.

To prove a stronger result than Proposition 2.10 in this situation, we need the following observation (cf., for example, [7, Lemma 3.4]).

Lemma 2.16. *Let M be a manifold and let Ω be a not necessarily closed differential form on M. For all $m \geq 1$ and all vector fields $v_1, \ldots, v_m$ in the Lie algebra $\mathfrak{X}(M)$,*

$$\begin{aligned}(-1)^m d\iota(v_1 \wedge \cdots \wedge v_m)\Omega &= \iota(\partial(v_1 \wedge \cdots \wedge v_m))\Omega\\ &\quad + \sum_{1=1}^{m} (-1)^i \iota(v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_m)\mathcal{L}_{v_i}\Omega + \iota(v_1 \wedge \cdots \wedge v_m) d\Omega.\end{aligned}$$

Definition 2.17. Given a differential form $\Omega \in \Omega^\bullet(M)$ and a multivector field $Y \in \Gamma(\Lambda^m TM)$, the *Lie derivative of* Ω *along* Y is defined, as a graded commutator, by $\mathcal{L}_Y\Omega := d\iota_Y\Omega - (-1)^m\iota_Y d\Omega$.

Remark 2.18. This definition allows us to combine the first and last terms in the above formula into a Lie derivative. Hence, the above formula can be written $\mathcal{L}_{v_1\wedge\cdots\wedge v_m}\Omega = (-1)^m[\iota(\partial(v_1 \wedge \cdots \wedge v_m))\Omega + \sum_{1=1}^{m}(-1)^i\iota(v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_m)\mathcal{L}_{v_i}\Omega]$.

Theorem 2.19. *Let $p \in Z_k(\mathfrak{g})$. Then $f_k(p)$ is a globally conserved quantity.*

Proof. We have

$$\iota_{v_H} df_k(p) = -\varsigma(k)\iota_{v_H}\iota(v_p)\omega = (-1)^k\varsigma(k)\iota(v_p)dH = \varsigma(k)d(\iota(v_p)H),$$

where we used (2.1) in the first equality and in the last equality Lemma 2.16 applied to the form H, as well as the $\mathfrak{g}$-invariance of H and once more the assumption that $p \in Z_k(\mathfrak{g})$.

Therefore, we conclude from Cartan's formula that

$$\mathcal{L}_{v_H} f_k(p) = d(\iota_{v_H} f_k(p) + \varsigma(k)\iota(v_p)H). \qquad \square$$

Remark 2.20. In the symplectic case, a homotopy co-momentum map boils down to its first component, $f_1 : \mathfrak{g} \to \Omega^0(M)$, a classical co-momentum map. Upon observing that $Z_1(\mathfrak{g}) = \mathfrak{g}$, the preceding theorem then reduces to the obvious but important fact that if a Hamiltonian function H is $\mathfrak{g}$-invariant, then for all x in $\mathfrak{g}$, we have that $\{f_1(x), H\} = \mathcal{L}_{v_H} f_1(x) = 0$.

In particular, by Theorem 2.19, $f_1(x)$ is a globally conserved quantity for all $x \in \mathfrak{g}$. Even in the case at hand of strictly H-preserving actions, $f_1(x)$ is not strictly conserved in general, as the following example shows.

Example 2.21. Let $M = \mathbb{R}^3$, $\omega = dx \wedge dy \wedge dz$, and $H = -x\,dy$. Then $v_H = \partial/\partial z$. Furthermore, we consider $\alpha = z dx$ and the $\mathbb{R}$-action given by $\mathfrak{g} = \mathbb{R} \to \mathfrak{X}(M)$, $1 \mapsto v_\alpha = -(\partial/\partial y)$. This action clearly admits a co-momentum map determined by $f_1(1) = \alpha$. Then $\mathcal{L}_{v_\alpha} H = 0$ but $\mathcal{L}_{v_H}\alpha = dx \neq 0$.

More is true: even if one assumes that $x \in B_1(\mathfrak{g})$ is a boundary, $f_1(x)$ is still not strictly conserved in general, as Example 3.19 below shows.

Specialising k to n in Theorem 2.19, we obtain scalar functions $f_n(x_1, \ldots, x_n)$ on M. Assembling these functions, we obtain a map $M \to Z_n(\mathfrak{g})^*$, very similar to the multimomentum maps of Madsen–Swann, cf. [7], except that it is not equivariant in general. As in the symplectic case, it satisfies the following corollary.

Corollary 2.22. *The vector field v_H is tangent to the level sets of the map $M \xrightarrow{\phi} Z_n(\mathfrak{g})^*$ given by $\phi(m)(p) = f_n(p)(m)$.*

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PROOF. We have to show that $v_H(m) \in \ker(T_m\phi : T_mM \to T_{\phi(m)}Z_n(\mathfrak{g})^* = Z_n(\mathfrak{g})^*)$. We have

$$\begin{aligned}((T_m\phi)(v_H(m)))(p) &= (df_n(p))(v_H(m)) \\ &= (\iota_{v_H} df_n(p))(m) = (\mathcal{L}_{v_H} f_n(p))(m) = 0,\end{aligned}$$

where the last equation uses the fact that $\mathcal{L}_{v_H} f_n(p)$ is exact and an exact function is necessarily 0. □

REMARK 2.23. An analogue of this result, where $Z_n(\mathfrak{g})$ is substituted by $B_n(\mathfrak{g})$, holds in the setting of Proposition 2.12.

We close this subsection by showing how a co-momentum map yields elements of the algebra $\mathcal{A}(v_H)$ from Lemma 1.15.

LEMMA 2.24. *For a strictly H-preserving infinitesimal action $x \mapsto v_x$:*

(i) *let $p \in Z_k(\mathfrak{g})$ for $k \geq 1$. Then $\iota(v_p)\omega \in \mathcal{A}(v_H)$;*
(ii) *let (f) be a co-momentum map for the $\mathfrak{g}$-action. Let $p \in Z_{k-1}(\mathfrak{g})$ for $k \geq 2$. Then $l_k(f_1(x_1), \ldots, f_1(x_{k-1}), H) \in \mathcal{A}(v_H)$.*

PROOF. Let us first observe that if $\alpha \in C(v_H)$, then $d\alpha \in \mathcal{A}(v_H)$. In fact, $d\alpha$, being exact, is closed. Furthermore, $\mathcal{L}_{v_H}(d\alpha) = d(\mathcal{L}_{v_H}\alpha) = 0$ since $\mathcal{L}_{v_H}\alpha$ is zero.

(i) By Theorem 2.19, $f_k(p)$ is a globally conserved quantity. As $\iota(v_p)\omega = \pm df_k(p)$ due to (2.1), it is an element of $\mathcal{A}(v_H)$ because of the preceding observation.
(ii) By Proposition 1.16, $l_k(f_1(x_1), \ldots, f_1(x_{k-1}), H)$ is conserved. We compute

$$\begin{aligned}d(\iota(v_1 \wedge \cdots \wedge v_{k-1})H) &= (-1)^{k-1}(\iota(v_1 \wedge \cdots \wedge v_{k-1})dH) \\ &= -(\iota(v_1 \wedge \cdots \wedge v_{k-1} \wedge v_H)\omega) = -\varsigma(k) l_k(f_1(x_1), \ldots, f_1(x_{k-1}), H),\end{aligned}$$

so $l_k(f_1(x_1), \ldots, f_1(x_{k-1}), H)$ is exact and in particular closed. □

2.5. The homological point of view. In this subsection we rephrase the 'generation of conserved quantities' via a co-momentum in a homological fashion.

Let $\mathfrak{g}$ be a Lie algebra acting on a pre-n-plectic manifold (M, ω), let H be a Hamiltonian $(n-1)$-form, and let v_H be a Hamiltonian vector field of H. Assume that the action is locally H-preserving, that is, $\mathcal{L}_{v_x}H$ is closed for all $x \in \mathfrak{g}$. The map $\mathfrak{g} \to H^{n-1}_{dR}(M), x \mapsto [\mathcal{L}_{v_x}H]$ measures how far the action is from being globally H-preserving. This map is 0 on $[\mathfrak{g}, \mathfrak{g}]$ and can thus be defined on $H_1(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Furthermore, it can be extended to a map on the whole Lie algebra homology.

PROPOSITION 2.25. *For every $k = 1, \ldots, \dim(\mathfrak{g})$, the map*

$$A : H_k(\mathfrak{g}) \to H^{n-k}_{dR}(M), \quad [p] \mapsto [\mathcal{L}_{v_p}H]$$

is well defined.

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Proof. Let $p \in Z_k(\mathfrak{g})$. We first check that $\mathcal{L}_{v_p} H$ is closed: putting $v_p = \sum_l v_1^l \wedge \cdots \wedge v_k^l$,

$$\begin{aligned} d\mathcal{L}_{v_p} H &= (-1)^{k+1} \mathcal{L}_{v_p} dH \\ &= -\Big(\iota(v_{\partial p}) dH + \sum_l \sum_{i=1}^{k} (-1)^i \iota(v_1^l \wedge \cdots \wedge \hat{v}_i^l \wedge \cdots \wedge v_m^l) \mathcal{L}_{v_i^l} dH\Big) = 0, \end{aligned}$$

where the first equality follows from Definition 2.17, the second one from Remark 2.18, and the last one from $\partial p = 0$ and the closedness of $\mathcal{L}_{v_i^l} H$.

Let $q \in \Lambda^{k+1}\mathfrak{g}$. Similarly to the above, we write $v_q = \sum_l v_1^l \wedge \cdots \wedge v_{k+1}^l$. We check that $\mathcal{L}_{v_{\partial q}} H$ is exact. By the definition of Lie derivative, this follows since

$$\begin{aligned} \iota(v_{\partial q}) dH &= (-1)^{k+1} \mathcal{L}_{v_q} dH - \sum_{i=1}^{k+1} \sum_l (-1)^i \iota(v_1^l \wedge \cdots \wedge \hat{v}_i^l \wedge \cdots \wedge v_{k+1}^l) \mathcal{L}_{v_i^l} dH \\ &= -d\mathcal{L}_{v_q} H \end{aligned}$$

is exact. Again, here in the first equality we used Remark 2.18 and in the second that $\mathcal{L}_{v_i^l} H$ is closed since the action is locally H-preserving. □

Remark 2.26.

(1) If the action is globally H-preserving, the map $\mathfrak{g} \to H^{n-1}_{dR}(M)$ is zero, but the higher components of A do not necessarily vanish. This is exhibited by Example 2.13: $\iota(v_a \wedge v_b) dH = -\iota(\partial/\partial x \wedge \partial/\partial y)(dx \wedge dy) = -1$ is closed but not exact.

(2) If the action is strictly H-preserving, then the map A is identically zero. Indeed, for every $p \in Z_k(\mathfrak{g})$, we have $\mathcal{L}_{v_p} H = 0$, as can be seen applying Lemma 2.16 to H.

When a co-momentum map exists, we can be more explicit.

Lemma 2.27. *If (f) is a co-momentum map for the $\mathfrak{g}$-action, then the map A can be written as follows: for all $p \in Z_k(\mathfrak{g})$,*

$$A([p]) = -\varsigma(k)[\mathcal{L}_{v_H} f_k(p)].$$

Proof. Let $p \in Z_k(\mathfrak{g})$. We have $A([p]) = [\mathcal{L}_{v_p} H] = (-1)^k [\iota(v_p)\iota_{v_H}\omega]$ using the definition of Lie derivative for multivector fields (see Remark 2.18). We can express this in terms of the co-momentum map using

$$\begin{aligned} (-1)^k \iota(v_p)\iota_{v_H}\omega &= \iota_{v_H}\iota(v_p)\omega \\ &= -\varsigma(k)\iota_{v_H} df_k(p) = -\varsigma(k)(-d\iota_{v_H} f_k(p) + \mathcal{L}_{v_H} f_k(p)). \end{aligned}$$

Passing to the cohomology class finishes the proof. □

Lemma 2.27 has several consequences, allowing us to recover some of our previous statements.

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REMARK 2.28.

(1) The form $f_k(p)$ is locally conserved if $p \in Z_k(\mathfrak{g})$ and globally conserved if $p \in B_k(\mathfrak{g})$. Hence, we recover Propositions 2.10 and 2.12.

(2) There is a canonical injective map $J : C_{\mathrm{loc}}(v_H)/C(v_H) \hookrightarrow H_{dR}(M), [\alpha] \mapsto [\mathcal{L}_{v_H}\alpha]$, as follows immediately from the definitions. The map A factors as

$$H_k(\mathfrak{g}) \longrightarrow \frac{C^{n-k}_{\mathrm{loc}}(v_H)}{C^{n-k}(v_H)} \xrightarrow{J} H^{n-k}_{dR}(M)$$

for every k, where the first map is induced by f_k multiplied by $-\varsigma(k)$. In particular, the map A takes values in the subspace $J(C_{\mathrm{loc}}(H)/C(H))$ of $H_{dR}(M)$.

(3) If the action is strictly H-preserving, by Remark 2.26(2), $f_k(p)$ is globally conserved for all $p \in Z_k(\mathfrak{g})$. Hence, we recover Theorem 2.19.

3. Examples and constructions

In the previous section we proved the existence of conserved quantities in the following set-up: G is a Lie group acting on a pre-n-plectic manifold (M, ω), $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$ is locally, globally, or strictly preserved, and (f) is a co-momentum map. Here we give several constructions that ensure the existence of a preserved Hamiltonian H and of a co-momentum map, in Sections 3.1 and 3.2. Further, in Section 3.3 we investigate actions which can be centrally extended by a Hamiltonian vector field and in Section 3.4 we construct an interesting class of examples for multisymplectic manifolds.

3.1. Constructing preserved Hamiltonians. In this subsection we consider a natural geometric situation in which the above machinery can be applied. We do not assume the existence of an invariant Hamiltonian form here, but we always assume a smooth action $\vartheta : M \times G \to M$ of a connected Lie group G on a pre-n-plectic manifold (M, ω) such that $\vartheta_g^*(\omega) = \omega$ for all $g \in G$.

LEMMA 3.1. *Consider a connected compact Lie group G acting on the pre-n-plectic manifold (M, ω) and let $\tilde{H} \in \Omega^{n-1}_{\mathrm{Ham}}(M)$, which is* locally *preserved by the action. Then there exists $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$ which is* strictly *preserved by the action and has the following property: any Hamiltonian vector field of $\tilde{H}$ is also a Hamiltonian vector field of H.*

PROOF. Define $H := \int_G \vartheta_g^*(\tilde{H})\mu(dg)$, the average of $\tilde{H}$ using the normalised Haar measure $\mu(dg)$ on G. Then H is strictly preserved by the action. Furthermore,

$$dH = \int_G \vartheta_g^*(d\tilde{H})\mu(dg) = d\tilde{H}.$$

The last equality holds because $\vartheta_g^*(d\tilde{H}) = d\tilde{H}$ for all $g \in G$, as a consequence of $d\mathcal{L}_{v_x}\tilde{H} = \mathcal{L}_{v_x} d\tilde{H} = 0$ for all $x \in \mathfrak{g}$. Hence, the Hamiltonian vector fields of $\tilde{H}$ are exactly the Hamiltonian vector fields of H. □

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Proposition 3.2. *Consider a connected compact Lie group G acting on the pre-n-plectic manifold (M, ω). Let v be a G-invariant vector field on M with $\mathcal{L}_v\omega = 0$. Suppose that $H^n_{dR}(M) = 0$. Then v is the Hamiltonian vector field of some Hamiltonian form H which is G-invariant, which is strictly preserved by the action.*

Proof. The condition $\mathcal{L}_v\omega = 0$ implies that $\iota_v\omega$ is closed and, by our cohomological assumption, it is exact:

$$\iota_v\omega = -d\tilde{H}$$

for some $\tilde{H} \in \Omega^{n-1}_{Ham}(M)$. Now we average both sides of the above equation. Notice that $\iota_v\omega$ is G-invariant, since v and ω are; hence, the averaged form $H := \int_G \vartheta^*_g(\tilde{H})\mu(dg)$ (where μ is the normalised Haar measure on G) satisfies the same equation: $\iota_v\omega = -dH$. □

An interesting special case is the one of volume forms.

Corollary 3.3. *Consider a connected compact Lie group G acting on (M, ω), where ω is a volume form. Let v be a G-invariant vector field on M which is divergence-free (that is, $\mathcal{L}_v\omega = 0$). Suppose that $H^{dim(M)-1}_{dR}(M) = 0$. Then v is the Hamiltonian vector field of some G-invariant Hamiltonian form H.*

Remark 3.4. If M is compact and simply connected, then $H^{dim(M)-1}_{dR}(M)$ vanishes by Poincaré duality and the above result is applicable.

The following statement is a variation of Proposition 3.2, in which the compactness assumption on G is replaced with the condition $H^{n-1}_{dR}(M) = 0$ and which leads to a globally preserved Hamiltonian.

Proposition 3.5. *Consider a connected Lie group G acting on the pre-n-plectic manifold (M, ω). Let v be a G-invariant vector field on M with $\mathcal{L}_v\omega = 0$. Suppose that $H^n_{dR}(M) = 0 = H^{n-1}_{dR}(M)$. Then v is the Hamiltonian vector field of some Hamiltonian form H which is globally preserved by the action.*

Proof. Since $\mathcal{L}_v\omega = 0$ and $H^n_{dR}(M) = 0$, we have $\iota_v\omega = -dH$ for some (usually not G-invariant) $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$. The form $\iota_v\omega$ is G-invariant, since v and ω are. Hence, for all $x \in \mathfrak{g}$, we have $0 = \mathcal{L}_{v_x} dH = d(\mathcal{L}_{v_x} H)$. The condition $H^{n-1}_{dR}(M) = 0$ implies that $\mathcal{L}_{v_x} H$ is exact. □

3.2. Induced actions of isotropy subgroups. Let G act on a pre-n-plectic manifold (M, ω). In this whole subsection we fix $p \in Z_k(\mathfrak{g}) \subset \Lambda^k\mathfrak{g}$ for some $k \geq 1$ (see Definition 2.8). We denote by G_p the corresponding isotropy group for the adjoint action of G on $\Lambda^k\mathfrak{g}$ and by $\mathfrak{g}_p$ its Lie algebra. Explicitly, $\mathfrak{g}_p = \{x \in \mathfrak{g} : [x, p] = 0\}$.

Remark 3.6. Let $p \in Z_k(\mathfrak{g})$ and $x \in \mathfrak{g}$. From Lemma 3.12, it follows that $x \wedge p \in Z_{k+1}(\mathfrak{g})$ iff $x \in \mathfrak{g}_p$.

Lemma 3.7. *The form $\iota(v_p)\omega \in \Omega^{n+1-k}(M)$ is closed and invariant under the action of G^0_p, the connected component of the identity in G_p.*

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Proof. The equality $d(\iota(v_p)\omega) = 0$ follows upon applying Lemma 2.16. The invariance holds since for every $y \in \mathfrak{g}_p$, $\mathcal{L}_{v_y}\iota(v_p)\omega = \iota([v_y, v_p])\omega + \iota(v_p)\mathcal{L}_{v_y}\omega = 0$, where the bracket is defined analogously to Lemma 3.12 later on. □

Now assume that there is a co-momentum map $(f) : \mathfrak{g} \to L_\infty(M, \omega)$.

Proposition 3.8. *A co-momentum map for the action of G_p^0 on $(M, \iota(v_p)\omega)$ is given by $(f^p) : \mathfrak{g}_p \to L_\infty(M, \iota(v_p)\omega)$ with components ($j = 1, \ldots, n-k$):*

$$f_j^p : \Lambda^j \mathfrak{g}_p \to \Omega^{n-k-j}(M),$$
$$q \mapsto -\varsigma(k) f_{j+k}(q \wedge p).$$

Furthermore, if the co-momentum map (f) is G-equivariant, then (f^p) is G_p^0-equivariant.

Proof. We first show that (f^p) is a co-momentum map. Let $q \in \Lambda^j \mathfrak{g}_p$. We have

$$\varsigma(k+j)\iota_{v_q}(\iota(v_p)\omega) = \varsigma(k+j)\iota(v_{p\wedge q})\omega = -f_{k+j-1}(\partial(p \wedge q)) - d(f_{k+j}(p \wedge q))$$

using in the second equality that (f) is a co-momentum map (see (2.1)). From Lemma 3.12, we obtain $\partial(p \wedge q) = (-1)^k p \wedge \partial(q) = (-1)^{kj}\partial(q) \wedge p$ since $p \in Z_k(\mathfrak{g})$ and $q \in \wedge^j \mathfrak{g}_p$. Using $\varsigma(k+j) = \varsigma(k)\varsigma(j)(-1)^{kj+1}$,

$$\varsigma(j)\iota_{v_q}(\iota(v_p)\omega) = -f_{j-1}^p(\partial(q)) - df_j^p(q).$$

For the equivariance statement, notice that for all $y \in \mathfrak{g}_p$,

$$\mathcal{L}_{v_y}(f_j^p(q)) = \mathcal{L}_{v_y}(f_{k+j}(p \wedge q)) = f_{k+j}([y, p \wedge q]) = f_{k+j}(p \wedge [y, q]) = f_j^p([y, q]),$$

where we used the equivariance of (f) in the second equality. □

Remark 3.9. The existence of a co-momentum map (f) implies that $\iota(v_p)\omega$ is exact with primitive $-\varsigma(k) f_k(p)$, by (2.1). Assume further that f_k is G-equivariant. Then this primitive is G_p^0-invariant for $\mathcal{L}_{v_y} f_k(p) = f_k([y, p]) = 0$ for all $y \in \mathfrak{g}_p$. Hence, by [2, Lemma 8.1], an equivariant co-momentum map for the action of G_p^0 on $(M, \iota(v_p)\omega)$ is given by ($j = 1, \ldots, n-k$)

$$\Lambda^j \mathfrak{g}_p \to \Omega^{n-k-j}(M),$$
$$q \mapsto (-1)^k \iota(v_q)(f_k(p)).$$

Notice that this co-momentum map may differ from the one given in Proposition 3.8.

Finally, we consider Hamiltonian forms.

Proposition 3.10. *Let $H \in \Omega_{\mathrm{Ham}}^{n-1}(M)$ be G-invariant; then $\iota(v_p)H$ is G_p^0-invariant and it is a Hamiltonian form with respect to $\iota(v_p)\omega$ with Hamiltonian vector field v_H.*

Proof. The G_p^0-invariance of $\iota(v_p)H$ is shown exactly as in Lemma 3.7. For the second statement, using Lemma 2.16, we compute

$$d(\iota(v_p)H) = (-1)^k \iota(v_p) dH = \iota_{X_H}(\iota(v_p)\omega).$$ □

Remark 3.11. Consider the case $k = n-1$. Then $\iota(v_p)\omega$ is a 2-form and, from Proposition 3.10, we recover the fact that $\iota(v_p)H$ is a conserved quantity (a special case of Proposition 3.15 later on).

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3.3. Co-momentum maps for $\mathfrak{g} \oplus \mathbb{R}$. We extend the results of Section 2.4 under a nondegeneracy assumption for ω. We assume that (M, ω) is an n-plectic manifold, $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$, and that $(f) : \mathfrak{g} \to L_\infty(M, \omega)$ is a co-momentum for a strictly H-preserving infinitesimal action $\mathfrak{g} \to \mathfrak{X}(M, \omega)$, $x \mapsto v_x$.

By Lemma 1.8(i), the generators of the action commute with the Hamiltonian vector field v_H of H, so the infinitesimal $\mathfrak{g}$-action on M extends to an action of the direct sum Lie algebra $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus \langle c \rangle_{\mathbb{R}}$, by means of $c \mapsto v_H$. Notice that $\Lambda^k \tilde{\mathfrak{g}} = \Lambda^k \mathfrak{g} \oplus (\Lambda^{k-1}\mathfrak{g} \otimes \langle c \rangle_{\mathbb{R}})$.

In the sequel we will make use of the following lemma several times. Recall that the differential ∂ was defined in Definition 2.2.

Lemma 3.12. *Let $p \in \Lambda^k \mathfrak{g}$ and $q \in \Lambda^l \mathfrak{g}$. Then*

$$\partial(p \wedge q) = \partial(p) \wedge q + (-1)^k p \wedge \partial(q) + (-1)^k [p, q],$$

where $[x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l] = \sum (-1)^{i+j} [x_i, y_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge \hat{y}_j \wedge \cdots \wedge y_l$.

Proof. It is sufficient to prove the assertion for monomials $p = x_1 \wedge \cdots \wedge x_k$ and $q = x_{k+1} \wedge \cdots \wedge x_{k+l}$. In that case $\partial(p \wedge q)$ is given by a sum over indices i, j with $1 \leq i < j \leq k + l$. Splitting it into sums over $i < j \leq k$, $k < i < j$, and $i \leq k < j$ proves the assertion. □

Remark 3.13. The bracket $[\cdot, \cdot] : \Lambda^\bullet \mathfrak{g} \times \Lambda^\bullet \mathfrak{g} \to \Lambda^\bullet \mathfrak{g}$ defined above turns $\Lambda^\bullet \mathfrak{g}$ into a Gerstenhaber algebra.

Lemma 3.14. *There is a canonical extension of (f) to a co-momentum map $(\tilde{f})$ for the $\tilde{\mathfrak{g}}$-action, determined by*

$$\tilde{f}_k(x_1, \ldots, x_{k-1}, c) = \varsigma(k) \iota(v_{x_1} \wedge \cdots \wedge v_{x_{k-1}}) H \tag{3.1}$$

for all $k \geq 1$ and $x_1, \ldots, x_{k-1} \in \mathfrak{g}$.

Proof. We have to check that (2.1) is satisfied. Without loss of generality, assume that $p = x_1 \wedge \cdots \wedge x_{k-1} \in \wedge^{k-1}\mathfrak{g}$ and notice that $[x_i, c] = 0$ for all i implies that $\partial(p \otimes c) = (\partial p) \otimes c$, by Lemma 3.12. Using the definition of $\tilde{f}_k$, (2.1) applied to $p \otimes c$ reads

$$-\varsigma(k-1)\iota(v_{\partial p})H = \varsigma(k) d\iota(v_p)H + \varsigma(k)(-1)^{k-1}\iota(v_p)\iota_{v_H}\omega.$$

Using $\iota_{v_H}\omega = -dH$, we see that this equation is satisfied by Lemma 2.16, since H is $\mathfrak{g}$-invariant and using the identity $\varsigma(k)\varsigma(k-1) = (-1)^k$. □

Notice that, even when (f) is equivariant, $(\tilde{f})$ is not equivariant in general. For instance, $\mathcal{L}_{v_H} f_1(x)$ is usually different from $\tilde{f}_1([c, x]) = \tilde{f}_1(0) = 0$. In general, it cannot be made equivariant by an averaging procedure since the group $G \times \mathbb{R}$ integrating $\tilde{\mathfrak{g}}$ is noncompact.

If $(\tilde{f})$ is equivariant, one has strong consequences: $\mathcal{L}_{v_H} f_k(x_1, \ldots, x_k) = 0$ and in particular $f_k(x_1, \ldots, x_k)$ is a strictly conserved quantity for all $x_1 \wedge \cdots \wedge x_k \in \wedge^k \mathfrak{g}$.

If we assume that $\mathcal{L}_{v_H} H = 0$, then the $\tilde{\mathfrak{g}}$-action strictly preserves the Hamiltonian H and hence we can apply Theorem 2.19 to the $\tilde{\mathfrak{g}}$-action and obtain globally conserved quantities for v_H for all elements of $Z_k(\tilde{\mathfrak{g}})$. As the latter is isomorphic to $Z_k(\mathfrak{g}) \oplus (Z_{k-1}(\mathfrak{g}) \otimes \langle c \rangle_{\mathbb{R}})$, these globally conserved quantities are those we already know from Theorem 2.19, plus those arising from $Z_{k-1}(\mathfrak{g}) \otimes \langle c \rangle_{\mathbb{R}}$. Somewhat surprisingly, it turns out that the latter are globally conserved quantities even without the assumption that $\mathcal{L}_{v_H} H = 0$. This fact is not predicted by Proposition 2.10, which only ensures the existence of locally conserved quantities.

PROPOSITION 3.15. *Assume that (M, ω) is an n-plectic manifold, H in $\Omega^{n-1}_{\mathrm{Ham}}(M)$, and that $(f) : \mathfrak{g} \to L_\infty(M, \omega)$ is the co-momentum of a strictly H-preserving Lie algebra action. The $\tilde{f}_k(p \otimes c)$ as in (3.1) is a globally conserved quantity for all $p \in Z_{k-1}(\mathfrak{g})$.*

PROOF. Because of Cartan's formula, it suffices to show that $\iota_{v_H} d\tilde{f}_k(p \otimes c)$ is exact. We will show that it actually vanishes. By Lemma 3.14 and using (2.1),

$$\iota_{v_H} d\tilde{f}_k(p \otimes c) = \iota_{v_H}(-\tilde{f}_{k-1}(\partial(p \otimes c)) - \varsigma(k)\iota(v_{p\otimes c})\omega).$$

Applying Lemma 3.12 to the Lie algebra $\tilde{\mathfrak{g}}$, we see that $\partial(p \otimes c) = 0$. Further, $v_c = v_H$, so by the skew symmetry of ω we get $\iota_{v_H}\iota(v_{p\otimes c})\omega = 0$, which finishes the proof. □

3.4. The multisymplectic analogue of magnetic terms. In this subsection we explain how to generalise the well-known magnetic term from symplectic geometry to the multisymplectic situation (compare [3, Section 7] for this construction) and provide the example announced in Section 2.4.

CONSTRUCTION 3.16. Let N be a manifold and c a closed $(k+1)$-form on N. Denoting the canonical projection from $\Lambda^k T^*N \to N$ by π and the canonical k-form on $\Lambda^k T^*N$ by $-\theta$, the $(k+1)$-form $\omega = d\theta + \pi^* c$ is always k-plectic, that is, nondegenerate and closed on $M = \Lambda^k T^*N$. The form $\pi^* c$ is called the *magnetic term*.

PROPOSITION 3.17. *Let $k \geq 1$ and N be a manifold, $b \in \Omega^k(N)$, and w a vector field on N such that $\mathcal{L}_w b = da$ for some $a \in \Omega^{k-1}(N)$ (that is, b is globally conserved by w). Denote the canonical lift of w to $M = \Lambda^k T^*N$ by w^h. Then w^h is a Hamiltonian vector field on $(M, \omega = d\theta + \pi^* db)$ with the following Hamiltonian $(k-1)$-form:*

$$H = -\pi^* a + \iota_{w^h}(\theta + \pi^* b). \tag{3.2}$$

PROOF. Upon observing that $\iota_{w^h}(\pi^* b) = \pi^*(\iota_w b)$ and consequently $\mathcal{L}_{w^h}(\pi^* b) = \pi^*(\mathcal{L}_w b)$,

$$\begin{aligned} dH &= -\pi^* \mathcal{L}_w b + d(\iota_{w^h}\theta) + \pi^*(d\iota_w b) \\ &= -\pi^* da - \iota_{w^h} d\theta + \mathcal{L}_{w^h}\theta - \pi^*(\iota_w db) + \pi^* \mathcal{L}_w b \\ &= -\pi^* da - \iota_{w^h} d\theta - \pi^*(\iota_w db) + \pi^* da = -\iota_{w^h}(d\theta + \pi^* db) \\ &= -\iota_{w^h}\omega, \end{aligned}$$

where in the third equality we used $\mathcal{L}_{w^h}\theta = 0$, as in the symplectic case. □

REMARK 3.18. If $N \times G \to N$ is a right action and b a G-invariant k-form on N, then the k-plectic form $\omega = d(\theta + \pi^* b)$ on $M = \Lambda^k T^*N$ has a G-invariant potential. This ensures the existence of a co-momentum map (see [2, Section 8.1]) whose first component $f_1 : \mathfrak{g} \to \Omega^{k-1}_{\mathrm{Ham}}(M)$ is given by $f_1(x) = \iota_{v_x^h}(\theta + \pi^* b)$.

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In Section 2.4 we announced an example of a strictly H-preserving action on a multisymplectic manifold admitting a co-momentum (f) such that, for some boundary $x \in B_1(\mathfrak{g})$, $f_1(x)$ is a globally conserved quantity that is *not strictly conserved*. We now provide this example.

EXAMPLE 3.19. Let $N \times G \to N$ be a right action and assume that, in the set-up of the preceding proposition, the vector field w and the forms a and b are G-invariant. Then H (see (3.2)) is invariant under the induced G-action on $M = \Lambda^k T^* N$.

Assuming furthermore that $db = 0$, we can choose $a := \iota_w b$. Specialise to $k = 2$ and $N = G$ with the action by $N \times G \to N, (n, g) \mapsto g^{-1} \cdot n$. Thus:

- for $x \in \mathfrak{g} = T_e G$ and $g \in G$, $v_x(g) = -(r_g)_*(x)$, where $r_g(h) = h \cdot g$ for $h \in G$. In particular, the generators v_x of the action are right-invariant vector fields on G;
- w is a left-invariant vector field, that is, there exists $\tilde{w} \in \mathfrak{g}$ such that for $g \in G$, $w(g) = (l_g)_*(\tilde{w})$, where $l_g(h) = g \cdot h$ for $h \in G$;
- b is a closed left-invariant 2-form, that is, there exists a $\tilde{b} \in \Lambda^2 \mathfrak{g}^*$ which is closed under the Chevalley–Eilenberg differential (the dual of the Lie algebra homology differential) and $b(g) = (l_{g^{-1}})^*(\tilde{b}) = ((l_{g^{-1}})_*)^*(\tilde{b})$ for all $g \in G$.

Denote by (f) the co-momentum map recalled in Remark 3.18. For any $x \in \mathfrak{g}$, we compute

$$\mathcal{L}_{w^h} f_1(x) = \iota_{v_x^h} \pi^*(\mathcal{L}_w b) = -\pi^*(d(\iota_{v_x} a)),$$

so $f_1(x)$ being a strictly conserved quantity is equivalent to $\iota_{v_x} a$ being a constant function on N. Evaluating the function $\iota_{v_x} a = b(w, v_x)$ at $g \in N = G$,

$$-\tilde{b}(\tilde{w}, Ad_{g^{-1}}(x)). \tag{3.3}$$

It is clear that the function (3.3) is not constant in general. For instance, take $G = SL_2(\mathbb{R})$. A basis for $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{R})$ is

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and $[h, e] = 2e, [h, f] = -2f, [e, f] = h$. So, notably, all elements of $\mathfrak{g}$ are boundaries. The form $\tilde{b} := e^* \wedge f^* \in \Lambda^2 \mathfrak{g}^*$ is closed (actually exact) with respect to the Chevalley–Eilenberg differential. Taking $\tilde{w} := f$ and $x := h \in \mathfrak{g} = B_1(\mathfrak{g})$, one computes that the function (3.3) attains the value $2\beta\delta$ at $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in G$ and hence it is not a constant function on G. We conclude that for this choice of $x \in B_1(\mathfrak{g})$, the form $f_1(x)$ is not strictly conserved.

4. Applications of conserved quantities

In Section 2 we saw that many conserved quantities exist on pre-n-plectic manifolds endowed with a co-momentum map. In this section we show some geometric consequences of the existence of conserved quantities on a manifold M, by looking at maps from a compact oriented manifold Σ into M. In most of our statements M does

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not need any additional geometric structure (but we specialise to the pre-n-plectic case for example in Proposition 4.9). In Section 4.1 we consider conserved quantities whose degree, as differential forms on M, equals $\dim(\Sigma)$. In Section 4.2 we extend some of the results to arbitrary degrees.

4.1. A general Kelvin circulation theorem. Let M be a manifold and $v \in \mathfrak{X}(M)$ a vector field. Let Σ be a compact, oriented d-dimensional manifold and $\sigma_0 : \Sigma \to M$ a smooth map. We view Σ as a 'membrane' in M, which evolves under the flow of the vector field, and want to find quantities which are unchanged under the evolution. The following theorem can be considered as folklore and can be viewed as a general version of Kelvin's circulation theorem, as we explain in Remark 4.6 below.

THEOREM 4.1. *Let Σ be a compact, oriented d-dimensional manifold (possibly with boundary), v a vector field on M with flow ϕ_t, and $\sigma_0 : \Sigma \to M$ a smooth map. Consider $\sigma_t := \phi_t \circ \sigma_0 : \Sigma \to M$. If $\alpha \in \Omega^d(M)$ is a differential form, then the number*

$$\int_\Sigma (\sigma_t)^* \alpha$$

is independent of the time parameter t if one of the following conditions holds:

(i) *α is strictly conserved by v;*

(ii) *α is globally conserved by v and Σ has no boundary;*

(iii) *α is locally conserved by v and there exists a compact, oriented manifold with boundary N such that $\Sigma = \partial N$ and a map $\tilde{\sigma}_0 : N \to M$ with $\tilde{\sigma}_0|_{\partial N} = \sigma_0$.*

REMARK 4.2. Since Σ is compact, there exists an $\varepsilon = \varepsilon(\sigma_0) > 0$ such that ϕ_t is defined at least on $(-\varepsilon, \varepsilon) \times \sigma_0(\Sigma) \subset \mathbb{R} \times M$. Obviously in (i) and (ii) we can consider $|t| < \varepsilon = \varepsilon(\sigma_0)$. *Mutatis mutandis*, we consider only $|t| < \varepsilon(\tilde{\sigma}_0)$ in case (iii). Notice that if $dim(M) = d$, or more generally if α is closed, then α is globally conserved.

PROOF. We only prove that condition (ii) suffices, for the other implications follow analogously. The diffeomorphisms ϕ_t satisfy $d/dt(\phi_t^* \alpha) = \phi_t^*(\mathcal{L}_v \alpha)$. Pre-composing with the pullback $(\sigma_0)^*$,

$$\frac{d}{dt}(\sigma_t^* \alpha) = \sigma_t^*(\mathcal{L}_v \alpha). \tag{4.1}$$

Hence, by compactness of Σ,

$$\frac{d}{dt} \int_\Sigma \sigma_t^* \alpha = \int_\Sigma \sigma_t^*(\mathcal{L}_v \alpha) = \int_\Sigma d(\sigma_t^* \gamma) = 0,$$

where in the first equality we used (4.1), in the second one that $\mathcal{L}_v \alpha = d\gamma$ for some form γ, and in the last one Stokes' theorem. □

REMARK 4.3. The sufficiency of Condition (ii) in Theorem 4.1 is not surprising. By assumption, v preserves α up to an exact form and, by Stokes' theorem, the contribution given by exact forms vanishes upon integration over Σ.

The following statement addresses a variation of condition (iii) in Theorem 4.1.

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PROPOSITION 4.4. *Let Σ be a compact, oriented manifold without boundary of dimension d and v a vector field on M with flow ϕ_t. If $\alpha \in \Omega^d(M)$ is locally conserved, then, for every fixed time t, one obtains a well-defined map*

$$F_t : [\Sigma, M] \to \mathbb{R}, \quad [\sigma_0] \mapsto \int_\Sigma (\sigma_t)^*\alpha - \int_\Sigma (\sigma_0)^*\alpha.$$

Here $[\Sigma, M]$ denotes the set of smooth homotopy classes of maps from Σ to M, $\sigma_0 : \Sigma \to M$ denotes a smooth map, and $\sigma_t := \phi_t \circ \sigma_0 : \Sigma \to M$.

Further, the dependence on t is linear: $F_t[\sigma_0] = t \cdot c([\sigma_0])$, where $c([\sigma_0]) := \int_\Sigma (\sigma_0)^*(\mathcal{L}_v\alpha)$.

PROOF. We have

$$\int_\Sigma (\sigma_t)^*\alpha - \int_\Sigma (\sigma_0)^*\alpha = \int_0^t \Big[\frac{d}{ds}\int_\Sigma (\sigma_s)^*\alpha\Big]\, ds = \int_0^t \Big[\int_\Sigma \sigma_s^*(\mathcal{L}_v\alpha)\Big]\, ds,$$

where the last equality is obtained as in the proof of Theorem 4.1. Now recall that $\mathcal{L}_v\alpha$ is a closed form on M. Hence, by Stokes' theorem, the term in the square bracket depends only on the homotopy class of σ_s, which agrees with the homotopy class of σ_0 since $\sigma_s = \phi_s \circ \sigma_0$. We conclude that the above expression equals $t \cdot c([\sigma_0])$. □

We present an example for Theorem 4.1 and Proposition 4.4.

EXAMPLE 4.5. Let M be a manifold, v a vector field, and $\alpha \in \Omega^d(M)$. Take a map $\sigma_0 : S^d \to M$ defined on the d-dimensional sphere and denote by σ_t the composition of σ_0 with the time-t flow ϕ_t of v. The number $\int_{S^d} (\sigma_t)^*\alpha$ is independent of the time parameter t if the following occurs: either (i) $\mathcal{L}_v\alpha$ is exact or (ii) $\mathcal{L}_v\alpha$ is closed and σ_0 is homotopy equivalent to a constant map. This follows from Theorem 4.1(ii) and (iii).

Further, assuming that $\mathcal{L}_v\alpha$ is closed, one obtains a well-defined group homomorphism

$$\pi_d(M, x) \to \mathbb{R}, \quad [\sigma_0] \mapsto \int_{S^d} (\sigma_t)^*\alpha - \int_{S^d} (\sigma_0)^*\alpha$$

defined on the dth homotopy group of M based at some point x and where the dependence on t is linear. This follows from Proposition 4.4 and the following argument to show the group homomorphism property. We denote the group multiplication of $\pi_d(M, x)$ by $*$. It is given by the following composition, where p denotes a distinguished point on the sphere:

$$f * g : (S^d, p) \to (S^d/S^{d-1}, p) = (S^d \vee S^d, p) \xrightarrow{(f \vee g)} (M, x).$$

Choosing appropriate representatives of the respective homotopy classes, we may assume that f, g, and $f * g$ are smooth. Then, for $\alpha \in \Omega^d(M)$, we calculate

$$\int_{S^d} (f * g)^*\alpha = \int_{S^d\setminus\{p\} \sqcup S^d\setminus\{p\}} (f \vee g)^*\alpha = \int_{S^d} f^*\alpha + \int_{S^d} g^*\alpha.$$

REMARK 4.6 (Kelvin circulation theorem). A variant of Theorem 4.1(ii) for a time-dependent vector field v^t and a time-dependent differential form $\alpha^t \in \Omega^d(M)$ is the following: if $\mathcal{L}_{v^t}\alpha^t + d/(dt)\alpha^t$ is exact and Σ has no boundary, then the number

$$\int_\Sigma (\sigma_t)^*(\alpha^t)$$

is independent of the time parameter t.

We mention this because *the Kelvin circulation theorem in fluid mechanics can be understood as a special case of the above*. Let $v^t = \sum_i v_i^t \partial_{x_i}$ be a time-dependent vector field on $\mathbb{R}^3$ and use the standard metric on $\mathbb{R}^3$ to obtain from v^t the 1-form $\alpha^t = \sum_i v_i^t dx_i$. One computes $\iota_{v^t} d\alpha^t = \sum_{i,k} v_k^t(\partial v_i^t/\partial x_k)dx_i - \frac{1}{2} d\sum_i (v_i^t)^2$, so that $\mathcal{L}_{v^t}\alpha^t + d/(dt)\alpha^t$ is exact if and only if

$$\sum_{i,k} v_k^t \frac{\partial v_i^t}{\partial x_k}\, dx_i + \sum_i \Big(\frac{d}{dt} v_i^t\Big)\, dx_i \tag{4.2}$$

is exact. By the above, it then follows that $\int_\Sigma (\sigma_t)^*\alpha^t$ is independent of t.

Upon rewriting the exactness of (4.2) as $(v^t \cdot \nabla)v^t + \partial v^t/\partial t = -\nabla w$, with ∇ the usual gradient in $\mathbb{R}^3$, we recognise the first of the isentropic Euler equations (see, for example, [4, page 15]). It is well known that this equation implies the classical Kelvin circulation theorem [4, page 21], which is exactly the time independence of $\int_\Sigma (\sigma_t^*)(\alpha^t)$ in this case.

We now bring multisymplectic forms into play. Under symmetry assumptions, our methods from Sections 2 and 3 lead to other conserved quantities, as we now explain. In fluid mechanics, the time-dependent vector field v^t is divergence free, that is, $\mathcal{L}_{v^t}\omega = 0$ for all t, where $\omega = dx_1 \wedge dx_2 \wedge dx_3$ is the standard volume form on $\mathbb{R}^3$. In the presence of a compact group of symmetries—that is, of an action of a compact Lie group G on $\mathbb{R}^3$ preserving ω and v^t—it turns out by Corollary 3.3 that, for every fixed value of t, the vector field v^t is the Hamiltonian vector field of a G-invariant Hamiltonian 1-form of $(\mathbb{R}^3, \omega)$. Further, ω is exact with G-invariant primitive, so that the action of G on $(\mathbb{R}^3, \omega)$ admits a co-momentum map [2, Section 8.1]. The latter, by virtue of Theorem 2.19, delivers further (time-independent) globally conserved quantities for v^t for a given value of t. One can then apply the above variant of Theorem 4.1(ii) to the time-dependent vector field v^t and to the newly obtained globally conserved quantities.

4.2. Transgression of conserved quantities. Theorem 4.1(ii) fits in the following framework. Let Σ be a compact, oriented manifold (without boundary) and M a manifold. Given a vector field v on M, there is a naturally associated vector field v^ℓ on $M^\Sigma = C^\infty(\Sigma, M)$, the space of smooth maps from Σ to M. It is given as follows:

$$v^\ell|_\sigma = \sigma^* v \in \Gamma(\sigma^* TM) = T_\sigma M^\Sigma$$

for all $\sigma \in M^\Sigma$. Notice that, denoting by ϕ_t the flow of v on M, the flow of v^ℓ maps $\sigma \in M^\Sigma$ to $\phi_t \circ \sigma \in M^\Sigma$. Similarly, associated to a differential form on M there is a

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differential form on M^Σ of lower degree. It is defined by the transgression map

$$\ell := \int_\Sigma \circ\, ev^* : \Omega^\bullet(M) \to \Omega^{\bullet - s}(M^\Sigma),$$

where $ev : \Sigma \times M^\Sigma \to M$ is the evaluation map and $\int_\Sigma$ denotes the integration along the fiber (cf., for example, [1, Ch. VI.4]) of the projection $\Sigma \times M^\Sigma \to M^\Sigma$.

Proposition 4.7. *Let Σ be a compact, oriented manifold (without boundary) of dimension d and let v be a vector field on M. If $\alpha \in \Omega^k(M)$ is globally (respectively locally respectively strictly) conserved by v, then $\alpha^\ell \in \Omega^{k-d}(M^\Sigma)$ is globally (respectively locally respectively strictly) conserved by v^ℓ.*

Proof. The transgression map ℓ commutes with de Rham differentials and furthermore we have $(\iota_v\alpha)^\ell = \iota_{v^\ell}\alpha^\ell$. Therefore, it commutes with Lie derivatives in the following sense: $\mathcal{L}_{v^\ell}\alpha^\ell = (\mathcal{L}_v\alpha)^\ell$. Assume that α is globally conserved. We have to show that $\mathcal{L}_{v^\ell}\alpha^\ell$ is an exact form. Since $\alpha \in \Omega^k(M)$ is a globally conserved quantity, there is a $\gamma \in \Omega^{k-1}(M)$ with $\mathcal{L}_v\alpha = d\gamma$. Hence,

$$\mathcal{L}_{v^\ell}\alpha^\ell = (\mathcal{L}_v\alpha)^\ell = (d\gamma)^\ell = d\gamma^\ell.$$

The other cases follow similarly. □

Remark 4.8. When $k = d$, Proposition 4.7 recovers Theorem 4.1(ii). Indeed, let $\alpha \in \Omega^d(M)$. By Proposition 4.7, the function α^ℓ on M^Σ is invariant under the flow of v^ℓ. The latter maps a point $\sigma \in M^\Sigma$ to $\phi_t \circ \sigma$, where ϕ_t denotes the flow of v on M. Finally, for all $\sigma \in M^\Sigma$,

$$\alpha^\ell|_\sigma = \left(\left(\int_\Sigma \circ\, ev^*\right)(\alpha)\right)\Big|_\sigma = \int_\Sigma (ev|_{\Sigma\times\{\sigma\}})^*\alpha = \int_\Sigma \sigma^*\alpha,$$

where in the second equality we used that $ev|_{\Sigma\times\{\sigma\}} = \sigma$.

Now we specialise to a pre-n-plectic manifold (M, ω) together with a vector field v_H which is Hamiltonian for some $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$. Notice that $(v_H)^\ell$ is a Hamiltonian vector field of H^ℓ on (M^Σ, ω^ℓ), as follows from

$$\iota_{(v_H)^\ell}\omega^\ell = (\iota_{v_H}\omega)^\ell = (-dH)^\ell = -dH^\ell.$$

A special case of Proposition 4.7 reads as follows.

Proposition 4.9. *Consider a pre-n-plectic manifold (M, ω) together with a vector field v_H which is Hamiltonian for some $H \in \Omega^{n-1}_{\mathrm{Ham}}(M)$. Let Σ be a compact, oriented manifold (without boundary) of dimension d. If $\alpha \in \Omega^k(M)$ is a globally conserved quantity for v_H, then*

$$\alpha^\ell \in \Omega^{k-d}(M^\Sigma)$$

is a globally conserved quantity for $(v_H)^\ell$, that is, $\mathcal{L}_{(v_H)^\ell}\alpha^\ell$ is an exact form.

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REMARK 4.10. Consider a G-action on (M, ω) for which H is strictly preserved. The G-action on M gives rise to a G-action on $M^\Sigma = C^\infty(\Sigma, M)$, simply given by $(g \cdot \sigma)(p) := g \cdot \sigma(p)$ for all $\sigma \in M^\Sigma$ and $p \in \Sigma$. It can be checked that for an infinitesimal generator v_x of the action on M ($x \in \mathfrak{g}$), the corresponding infinitesimal generator of the action on Σ is $(v_x)^\ell$. Hence, the lifted G-action preserves ω^ℓ and H^ℓ. In [2, Section 11] and [5, Section 6], it is shown that a co-momentum map for the action of G on (M, ω) transgresses to a co-momentum map for the action on (M^Σ, ω^ℓ). This is consistent with the fact that, firstly, certain components of co-momentum maps are globally conserved quantities (Theorem 2.19) and, secondly, globally conserved quantities transgress to give globally conserved quantities (Proposition 4.9).

Acknowledgements

M.Z. thanks Chris Rogers for constructive comments, especially during the first phase of the project, and Daniel Peralta Salas for making him aware of the Kelvin circulation theorem. M.Z. was partially supported by grants MTM2011-22612 and ICMAT Severo Ochoa SEV-2011-0087 (Spain), by Pesquisador Visitante Especial grant 88881.030367/2013-01 (CAPES/Brazil), by IAP Dygest (Belgium), and by the long term structural funding Methusalem grant of the Flemish Government. We thank the IECL Metz-Nancy (UMR 7502, France), KU Leuven, and Ruhr-University Bochum (via its Research School Plus, funded by Germany's Excellence Initiative [DFG GSC 98/3]) for making possible mutual visits of the authors.

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https://doi.org/10.1017/S1446788718000381 Published online by Cambridge University Press