1

Human

Problem 1 Human Eye

How do we see?
What kind of glasses might we need?
When can we distinguish between the two eyes of a cat during the night?

A schematic view of the structure of the human eye is presented in Figure 1.1. Light rays that refract at the cornea and eye lens end up at the retina, which produces nerve impulses sent to the brain down the optic nerve. In a simplified model of an eye, the cornea and eye lens can be replaced with one converging lens (called simply the lens in the remainder of the text) while the retina can be modeled as a disk of radius $R = 1.00\,\mathrm{cm}$, the axis of which coincides with the optical axis of

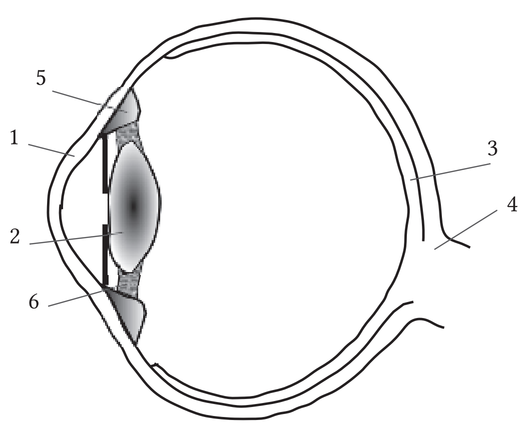

Figure 1.1 Scheme of the structure of the human eye: (1) cornea, (2) eye lens, (3) retina, (4) optic nerve, (5) ciliary muscles, (6) suspensory ligament

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the lens, as shown in Figure 1.2. The distance between the retina and the lens is $d = 2.40\,\text{cm}$. A human can adjust the focal length of the lens and therefore has the capability of clearly seeing objects at different distances. This process is called eye accommodation and is enabled by ciliary muscles connected to the eye lens by a suspensory ligament. These muscles act to tighten or relax the ligaments and therefore thin down or thicken the lens. Consequently the focal length of the lens changes.

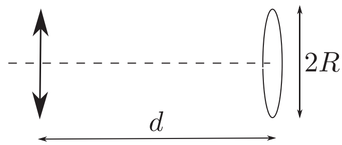

Figure 1.2 A simplified model of the human eye

(a) A human has regular eyesight if images of all objects from a distance larger than $d_0 = 25.0\,\text{cm}$ can be formed at the retina. What is the range of the lens' focal lengths for a human with regular eyesight?
(b) The maximal focal length f_{max} of the lens for a nearsighted man is smaller than the upper limit of the range determined in part (a). This man uses glasses with a diopter value of $D_1 = -1.00\,\text{m}^{-1}$ to clearly see very distant objects. Determine f_{max} and find the maximal distance of an object that this man can clearly see without using the glasses. For simplicity neglect the distance between the glasses and the lenses.
(c) The minimal focal length f_{min} of the lens for a farsighted woman is larger than the lower limit of the range determined in part (a). This woman needs glasses with a diopter value of $D_2 = 2.00\,\text{m}^{-1}$ to clearly see objects at a distance of $d_0 = 25.0\,\text{cm}$. Determine f_{min} and find the minimal distance of an object that this woman can clearly see without using the glasses.
(d) A person is nearsighted (farsighted) as well when the distance between the retina and the lens is larger (smaller) than the regular distance of $d = 2.40\,\text{cm}$. Calculate the diopter value of the glasses that should be used by a man with a distance between the retina and the lens of $d_1 = 2.50\,\text{cm}$ ($d_2 = 2.30\,\text{cm}$).

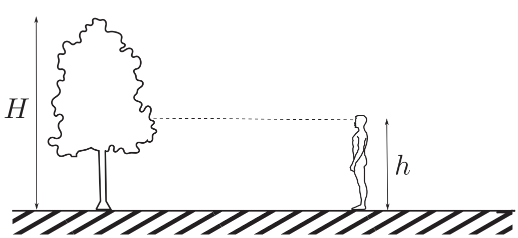

Figure 1.3 With problem 1(e)

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(e) A man with regular eyesight whose height is $h = 2.00\,\mathrm{m}$ is observing a tree of height $H = 2h$ (Figure 1.3). His view is directed toward the middle of the tree. What is the minimal distance between the man and the tree that allows him to see the whole tree?

Two types of light receptors are placed at the retina – rods (about $N_1 = 10^8$ of them) and cones (about $N_2 = 6 \cdot 10^6$ of them). Rods enable night vision, while cones are used for vision during the day. Assume that a person can distinguish two distant objects during the day (night) if their images are at different cones (rods). Assume also that the cones (rods) are evenly distributed on the retina surface and that their positions form a square lattice.

(f) Two point objects are at a mutual distance of $a = 1.00\,\mathrm{mm}$. The direction that connects them is perpendicular to the optical axis of the lens (Figure 1.4). What is the maximal distance from which a woman can distinguish between these two objects during the day?

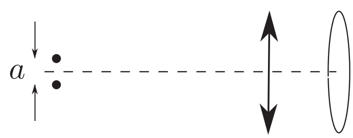

Figure 1.4 With problem 1(f)

(g) At what maximal distance can a woman read the license plates of a car during the day? Assume that the license plates can be read if a woman can distinguish between the point objects at a mutual distance of $a = 1.00\,\mathrm{cm}$.

(h) At what maximal distance can a woman distinguish between the two eyes of a cat during the night? The eyes of a cat are at a mutual distance of $a = 2.00\,\mathrm{cm}$.

Solution of Problem 1

(a) To see an object at a distance p from the eye, a human needs to accommodate the focal length of the lens so that the image of the object is formed at the retina (which is at a distance $l = d$ from the lens). For an object at a distance $p_1 = d_0$ the focal length is given by lens equation $\frac{1}{f_1} = \frac{1}{p_1} + \frac{1}{l}$. For an object at a distance $p_2 \to \infty$ we have $\frac{1}{f_2} = \frac{1}{p_2} + \frac{1}{l}$. From previous equations we obtain $f_1 = 2.19\,\mathrm{cm}$ and $f_2 = 2.40\,\mathrm{cm}$. Consequently the lens focal length of a human with regular eyesight takes a range from $f_1 = 2.19\,\mathrm{cm}$ to $f_2 = 2.40\,\mathrm{cm}$.

(b) The lens focal length and the distance of the object that the man clearly sees are related by $\frac{1}{f} = \frac{1}{p} + \frac{1}{d}$. Consequently, without the use of glasses, this man cannot clearly see objects at a distance larger than $p_{\max}$, where

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$$\frac{1}{f_{\max}} = \frac{1}{p_{\max}} + \frac{1}{d}. \tag{1.1}$$

The focal length of the system lenses-glasses f_{ns} satisfies the relation $\frac{1}{f_{\mathrm{ns}}} = \frac{1}{f} + D_1$. When this man clearly sees very distant objects with the use of glasses, the lens equation reads

$$\frac{1}{f_{\max}} + D_1 = \frac{1}{p_2} + \frac{1}{d}, \tag{1.2}$$

where $p_2 \to \infty$. From equation (1.2) we obtain $f_{\max} = \frac{d}{1-dD_1} = 2.34\,\mathrm{cm}$. By subtracting equations (1.1) and (1.2) we find $p_{\max} = -\frac{1}{D_1} = 1.00\,\mathrm{m}$.

(c) Without the use of glasses, this woman cannot clearly see objects at a distance smaller than $p_{\min}$, where

$$\frac{1}{f_{\min}} = \frac{1}{p_{\min}} + \frac{1}{d}. \tag{1.3}$$

The lens equation for a woman with glasses looking at an object at a distance d_0 reads

$$\frac{1}{f_{\min}} + D_2 = \frac{1}{d_0} + \frac{1}{d}. \tag{1.4}$$

From equation (1.4) it follows that

$$f_{\min} = \frac{1}{\frac{1}{d_0} + \frac{1}{d} - D_2} = 2.29\,\mathrm{cm}. \tag{1.5}$$

By subtracting equations (1.3) and (1.4) we obtain

$$p_{\min} = \frac{d_0}{1 - D_2 d_0} = 50.0\,\mathrm{cm}. \tag{1.6}$$

(d) The lens equation for a man with regular distance between the lens and the retina when he clearly sees an object at a distance p is $\frac{1}{f} = \frac{1}{p} + \frac{1}{d}$. For a man with distance d_i between the retina and the lens who uses glasses with diopter value D_i and clearly sees the same object when the lens focal length is the same, we obtain $\frac{1}{f} + D_i = \frac{1}{p} + \frac{1}{d_i}$. Subtracting the previous two equations, we find $D_i = \frac{1}{d_i} - \frac{1}{d}$. Consequently, we find in the first case $D_1 = -1.67\,\mathrm{m}^{-1}$ and in the second case $D_2 = 1.81\,\mathrm{m}^{-1}$.

(e) A man sees the whole tree when the size L of the image of the tree on the retina is smaller than the retina diameter (Figure 1.5). Using the similarity of the triangles in Figure 1.5 we obtain $\frac{L}{H} = \frac{d}{x}$, where x is the distance between the man and the tree. Consequently the man sees the whole tree when $L = H\frac{d}{x} < 2R$, leading to $x > \frac{Hd}{2R} = 4.80\,\mathrm{m}$.

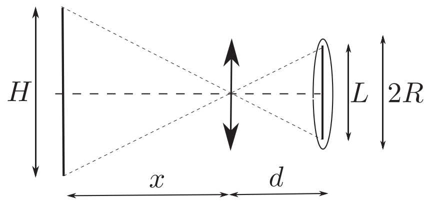

Figure 1.5 With the solution of problem 1(e)

(f) The number of cones per unit surface is equal to $N_S = \frac{N_2}{R^2\pi}$. On the other hand, since we assume that the positions of cones form a square lattice with lattice constant b, we also have $N_S = \frac{1}{b^2}$. From the previous two equations it follows that $b = R\sqrt{\frac{\pi}{N_2}} = 7.24\,\mu$m. When the woman is at a maximal distance at which she can still distinguish between the two objects, the images of the objects are formed at two neighboring cones. From the similarity of triangles in Figure 1.6, we find $\frac{a}{x} = \frac{b}{d}$ – that is, $x = \frac{ad}{b} = 3.32\,$m.

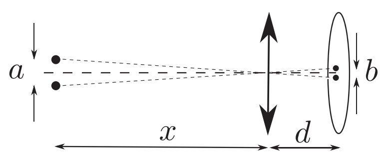

Figure 1.6 With the solution of problem 1(f)

(g) From the solution of part (f) we have $x = \frac{ad}{b}$, where in this case $a = 1.00\,$cm, leading to $x = 33.2\,$m.

(h) Since the woman observes the cat during the night, the solution of part (f) is modified only by replacing the number of cones with the number of rods. Consequently, $x = \frac{ad\sqrt{N_1}}{R\sqrt{\pi}} = 271\,$m.

We refer the reader interested in more details regarding the physics of the human eye to chapter 12, reference [13].

Problem 2 The Circulation of Blood

How powerful is the human heart?
How does a bypass help in the case of arteriosclerosis?

The human cardiovascular system consists of the heart, the blood, and the blood vessels. The heart pumps the blood through the blood vessels. The blood carries nutrients and oxygen to and carbon dioxide away from various organs. The most

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important portions of the cardiovascular system are pulmonary circulation and systemic circulation. Pulmonary circulation pumps away oxygen-depleted blood from the heart via the pulmonary artery to the lungs. It then returns oxygenated blood to the heart via the pulmonary vein. Systemic circulation transports oxygenated blood away from the heart through the aorta. The aorta branches to arteries that bring the blood to the head, the body, and the extremities. The veins then return oxygen-depleted blood to the heart. The direction of blood flow is determined by four heart valves. Two of them are positioned between the antechambers and the chambers, while two are located between the chambers and the arteries.

(a) The heart pumps blood by contraction of the muscles of the antechambers and chambers. The blood pressure gradually increases from the minimal (diastolic) value of $p_d = 80\,\mathrm{mmHg}$ to the maximal (systolic) value of $p_s = 120\,\mathrm{mmHg}$ during contraction.

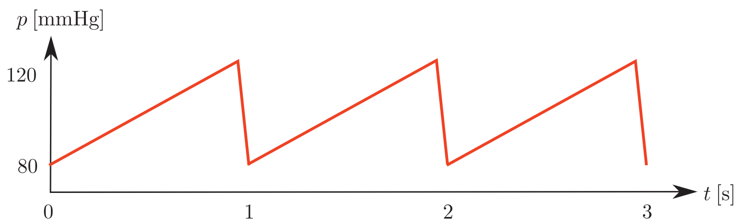

Figure 1.7 The graph of the dependence $p(t)$

The muscle then relaxes and the value of pressure suddenly decreases, as shown in Figure 1.7. The heart contracts (beats) around 60 times a minute. Each contraction pumps around 75 ml of blood. The pump shown in Figure 1.8 is a simple model of the heart. The heart decreases the volume during the contraction, which corresponds to the upward motion of the piston in the model.

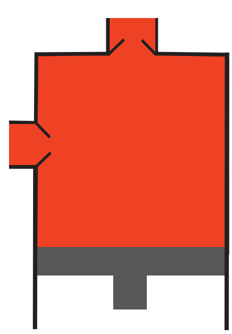

Figure 1.8 A pump as a model of the heart

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Thereby the pressure increases and closes the input valves while it opens the output valves. Determine the power of the heart.

The boundary between laminar and turbulent flow of blood is determined from the Reynolds number, which is directly proportional to the speed of blood v. The Reynolds number is a dimensionless quantity that depends as well on the density of blood $\rho = 1{,}060\,\mathrm{kg/m^3}$, viscosity of blood $\eta = 4.0 \cdot 10^{-3}\,\mathrm{Pa \cdot s}$ and the diameter of the blood vessel D. The flow is turbulent if the Reynolds number is larger than 2,000, while it is laminar otherwise.

(b) Derive the expression for the Reynolds number using dimensional analysis. Assume that the dimensionless constant that appears in front of the expression is equal to 1.
(c) The diameter of the aorta is $D = 10\,\mathrm{mm}$. Calculate the maximal speed of laminar blood flow in the aorta.

We consider next the laminar flow of blood through the artery whose shape is a cylinder of length L and radius R, as shown in Figure 1.9. The flow of blood in the artery is caused by the difference of pressures Δp at the ends of the artery, which is a consequence of blood pumping from the heart. The blood does not slide at the walls of the artery. For this reason, a cylindrical layer of blood that is at rest is formed near the wall of the artery. The viscosity of the blood causes laminar flow where each layer slides between neighboring layers. The viscosity force between the layers F is given by Newton's law,

$$F = \eta S \frac{\Delta v}{\Delta r},$$

where η is the viscosity of the blood, S is the area of the layer that is in contact with the neighboring layer, and $\Delta v / \Delta r$ is the gradient of speed in the radial direction. The walls of the artery are inelastic and the speed of flow does not change between the points on the same line in the direction of the artery.

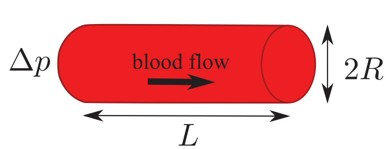

Figure 1.9 Artery

(d) Determine the dependence of the speed of blood on the distance from the artery axis.
(e) Using the analogy of electrical resistance, one can define the resistance of blood flow as the ratio of the pressure difference and the volume flow caused by this difference of pressures. Determine the blood flow resistance through the artery.

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(f) As a consequence of arteriosclerosis, the inner diameter of a part of the artery decreased from $d_1 = 6.0\,\mathrm{mm}$ to $d_2 = 4.0\,\mathrm{mm}$. How many times was the blood flow resistance increased in this part of the artery? To reduce the blood flow resistance, a bypass can be introduced. A healthy artery or vein is removed from another part of the patient's body and attached in parallel to this part of the artery. Assume that the bypass is of the same length as this part of the artery. How many times does the blood flow resistance decrease after the introduction of a bypass of diameter $d_3 = 5.0\,\mathrm{mm}$?

When the blood enters the artery, the speed of the blood is nearly the same throughout the cross-section of the artery. This means that the blood needs to accelerate and decelerate to reach the regime considered in previous parts of the problem. The blood near the artery walls decelerates to zero speed, while the part in the center of the artery accelerates to the maximal value of the speed. Consider the situation when we neglect the viscosity and when the blood accelerates along the artery.

(g) Determine the relation between the pressure difference Δp at the ends of the artery and the change of volume flow $\Delta q/\Delta t$ as a function of blood density ρ, the length of the artery L, and its radius R.
(h) As in part (e), the analogy with electrical circuits can be also introduced in part (g). Which element of the electric circuit can be used to describe the relation determined in part (g)?

Solution of Problem 2

(a) The work performed by the pump when the piston moves by Δr is

$$\Delta A = F\Delta r = \frac{F}{S}\Delta r S = p\Delta V\,. \tag{1.7}$$

The work performed by the heart is equal to the area under the graph of the function $p(V)$. The heart performs 60 beats per minute, which is 1 beat per second. Consequently the heart pumps in $V = 75\,\mathrm{ml}$ of blood each second. Therefore, the graph of the function $p(V)$ looks as shown in Figure 1.10. The work performed by the heart during 1 beat is

$$A = p_d V + \frac{1}{2}(p_s - p_d)V = \frac{1}{2}(p_s + p_d)V = 1.0\,\mathrm{J}\,. \tag{1.8}$$

The work A is performed by the heart during $t = 1\,\mathrm{s}$, which means that the corresponding power is $P = A/t = 1\,\mathrm{J}/1\,\mathrm{s} = 1.0\,\mathrm{W}$.

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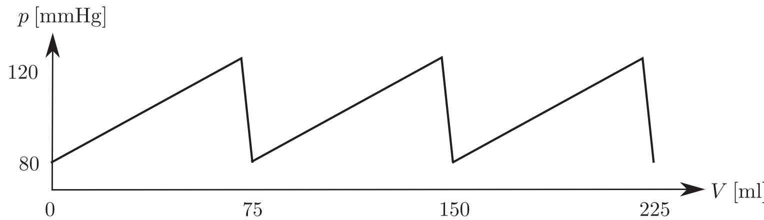

Figure 1.10 The graph of the function $p(V)$

(b) We can find the expression for the Reynolds number using dimensional analysis

$$\mathrm{Re} = v\rho^{\alpha}\,\eta^{\beta}\,D^{\gamma} \Rightarrow 1 = [\mathrm{m}\cdot\mathrm{s}^{-1}]\,[\mathrm{kg}\cdot\mathrm{m}^{-3}]^{\alpha}\,[\mathrm{kg}\cdot\mathrm{m}^{-1}\mathrm{s}^{-1}]^{\beta}\,[\mathrm{m}]^{\gamma}, \tag{1.9}$$

which leads to the system of equations

$$1-3\alpha-\beta+\gamma = 0, -1-\beta = 0, \alpha+\beta = 0\,, \tag{1.10}$$

whose solution is $(\alpha,\beta,\gamma) = (1,-1,1)$. Therefore, the Reynolds number is given by the expression

$$\mathrm{Re} = \frac{\rho v D}{\eta}\,. \tag{1.11}$$

(c) The maximal speed of blood in the aorta is obtained for $\mathrm{Re} = 2{,}000$ and reads

$$v = \frac{\eta \mathrm{Re}}{\rho D} = 75\,\frac{\mathrm{cm}}{\mathrm{s}}\,. \tag{1.12}$$

The Reynolds number reaches the critical value when the valves of the aorta open. The blood is then under big pressure and reaches a speed as high as $120\,\mathrm{cm/s}$. So-called Korotkoff sounds appear then as a consequence of turbulent flow. These can be heard using a stethoscope. This fact is used when blood pressure is measured using a sphygmomanometer.

(d) The system has cylindrical symmetry. Consequently the speed of blood is constant in each thin cylindrical layer. Consider the part of blood in the shape of a cylinder of radius r. This part of blood in the artery moves due to pressure difference Δp, which yields the force $F_1 = \pi r^2 \Delta p$. The magnitude of the viscosity force that acts on this layer is $F_2 = 2\pi r L \eta \frac{\Delta v}{\Delta r}$. Since each layer of blood is moving at a constant velocity, we obtain from Newton's first law that

$$F_1 = F_2 \Rightarrow \Delta v = \frac{\Delta p}{2\eta L} r \Delta r\,. \tag{1.13}$$

By transforming the equation (1.13) to differential form and performing integration with the boundary condition $v(R) = 0$, we obtain the dependence of the speed of blood on the distance from the axis of the artery:

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$$v(r) = \frac{\Delta p}{4\eta L}(R^2 - r^2)\,. \tag{1.14}$$

(e) The flow of blood through the ring of width dr, which is located in the region between r and $r + dr$, is $v(r)dS$, where $dS = 2\pi r dr$ is the area of that ring. The flow of blood through the artery is then obtained by performing the integration over all rings, which leads to

$$q = \int_0^R v(r) 2\pi r dr = \int_0^R \frac{\pi \Delta p}{2\eta L}(rR^2 - r^3)dr = \frac{\pi \Delta p R^4}{8\eta L}\,, \tag{1.15}$$

and consequently the blood flow resistance is

$$\mathcal{R} = \frac{\Delta p}{q} = \frac{8\eta L}{\pi R^4}\,. \tag{1.16}$$

(f) Due to arteriosclerosis the blood flow resistance in the sick part of the artery $\mathcal{R}_2$ increases in comparison to the resistance in the healthy artery $\mathcal{R}_1$, which leads to

$$\frac{\mathcal{R}_2}{\mathcal{R}_1} = \left(\frac{d_1}{d_2}\right)^4 = 5.1\,. \tag{1.17}$$

After the bypass is introduced, the sick part of the artery and the bypass form a parallel connection of two resistors with equivalent resistance $\mathcal{R}_e$. The resistance then reduces by

$$\frac{\mathcal{R}_2}{\mathcal{R}_e} = \frac{\mathcal{R}_2}{\frac{\mathcal{R}_2\mathcal{R}_3}{\mathcal{R}_2+\mathcal{R}_3}} = 1 + \left(\frac{d_3}{d_2}\right)^4 = 3.4\,. \tag{1.18}$$

(g) Newton's second law applied to the blood in the artery gives:

$$m\frac{\Delta v}{\Delta t} = \Delta p S\,, \tag{1.19}$$

where $m = \rho V = \rho L \pi R^2$ is the mass of the blood in the artery, $\Delta v / \Delta t$ is the change of the speed of blood along the artery, and $S = \pi R^2$ is the area of the inner cross-section of the artery. The change of flow is $\Delta q = \Delta(R^2 \pi v) = \pi R^2 \Delta v$, which along with equation (1.19) gives

$$\Delta p = \left(\frac{\rho L}{\pi R^2}\right)\frac{\Delta q}{\Delta t}\,. \tag{1.20}$$

(h) One can conclude from part (e) that the change of pressure is analogous to the potential difference, while the flow of blood is analogous to the electrical current. Consequently equation (1.20) is analogous to the equation

$$\Delta \varphi = \mathcal{L}\frac{\Delta I}{\Delta t}\,, \tag{1.21}$$

which leads to the conclusion that the inductor is the analogous electrical component that describes the blood flow in part (g).

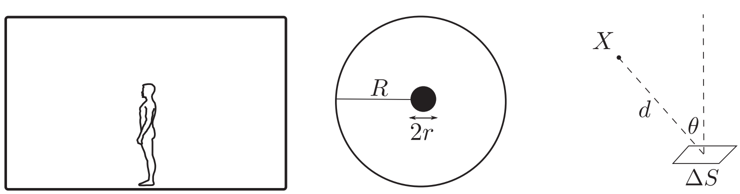
We refer the reader interested in more details about the circulation of blood in the human body to references [23] and [41].

Problem 3 A Human As a Heater

How many persons are needed to heat a room to the same temperature as a heater?

You will have certainly noticed that it can be very hot in a room where a lot of people are present. The reason for this is that humans emit heat and consequently they heat the room. We assume that humans exchange heat with the room only by thermal radiation and heat conduction.

We consider first how a human exchanges heat with their surroundings by thermal radiation. We use a simplified model in which the shape of the human is a ball of radius r, while the room is a sphere of radius R, which is much larger than r. The centers of the ball and the sphere coincide, as shown in Figure 1.11(b). Assume that the human and the internal walls of the room radiate as black bodies, whose temperatures are respectively T_c and T_z.



(a) (b) (c)

Figure 1.11 (a) A human in the room. (b) Simplified model of the human and the room. (c) A scheme accompanying equation (1.22)

To solve the problem you can make use of the following fact. In point X the intensity of electromagnetic radiation emitted by a black body in the shape of a small flat tile is given as

$$I = \frac{\sigma T^4 \Delta S \cos\theta}{d^2 \pi}, \tag{1.22}$$

where T is the temperature of the body, ΔS is the area of one side of the tile, d is the distance between the tile and point X, and θ is the angle between the direction connecting point X with the tile and the direction perpendicular to the plane of the tile, as shown in Figure 1.11(c).

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(a) Determine the expression for the power of the radiation that the human emits.
(b) Determine the expression for the power of the radiation that the human absorbs.
(c) Determine the expression for the power of the radiation that the human exchanges with their surroundings.
(d) Calculate the power of the radiation that the human exchanges with their surroundings. Assume that the equations derived in previous parts of the problem can be applied to a human in the room. Use the following numerical values in this part of the problem – $t_c = 36.0°\text{C}$, $t_z = 20.0°\text{C}$ – and assume that the surface area of one human is equal to $S_c = 1.90\,\text{m}^2$. The Stefan–Boltzmann constant is $\sigma = 5.67 \cdot 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}$, $0°\text{C} = 273.15\,\text{K}$.

The law of heat conduction states that the amount of heat that a human exchanges with their surroundings in unit time by heat conduction is given by the expression

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \alpha S_c (T_o - T_c), \tag{1.23}$$

where α is the coefficient of heat conduction equal to $\alpha = 4.50\,\frac{\text{W}}{\text{m}^2\cdot\text{K}}$ for a human wearing regular clothes, S_c is the surface area of the human, and T_o is the temperature of the surroundings.

(e) Calculate the power of heat that a human from part (d) exchanges with their surroundings by heat conduction. Assume that the temperature of the surroundings is equal to the temperature of walls from part (d).
(f) Calculate how many persons are needed to heat the room to the same temperature as the heater, whose useful power is $P_g = 1.30\,\text{kW}$. Assume that the formulas derived in previous parts of the problem can be applied to each person in the room and that people exchange heat with their surroundings only by radiation and conduction.
(g) Answer the same question as in part (f) in the case when people in the room wear winter clothes whose coefficient of heat conduction is $\alpha' = 2.40\,\frac{\text{W}}{\text{m}^2\cdot\text{K}}$.

Solution of Problem 3

(a) Since the human emits as a black body, the power of the emitted radiation is given by the Stefan–Boltzmann law $P_{\text{em}} = \sigma T_c^4 S_c$, where $S_c = 4r^2\pi$ is the surface area of the human, which leads to $P_{\text{em}} = \sigma T_c^4 4r^2\pi$.
(b) We determine first the power of the radiation emitted by a small part of the wall of area ΔS_z that is absorbed by the human. We denote this power as ΔP_{pr}. The intensity of the radiation emitted by that part of the wall at a distance R is, according to the formula given in the problem, $\Delta I = \frac{\sigma T_z^4 \Delta S_z}{R^2\pi}$. The radiation incident on the human is the radiation that would be incident on a tile of area $r^2\pi$ that is perpendicular to the direction that connects the part of the wall and

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the human. Therefore, $\Delta P_{\mathrm{pr}} = \Delta I \cdot r^2 \pi$, which leads to $\Delta P_{\mathrm{pr}} = \frac{\sigma T_z^4 \Delta S_z r^2}{R^2}$. The total power of the radiation absorbed by the human is obtained by adding all ΔP_{pr}. Bearing in mind that $S_z = 4R^2\pi$, we obtain $P_{\mathrm{pr}} = \sigma T_z^4 4\pi r^2$.

(c) The power that the human exchanges with their surroundings is $P_{\mathrm{r}} = P_{\mathrm{em}} - P_{\mathrm{pr}} = \sigma\left(T_c^4 - T_z^4\right) 4\pi r^2$.

(d) Using the expression from part (c) leads to $P_{\mathrm{r}} = \sigma\left(T_c^4 - T_z^4\right) S_c$, and consequently $P_{\mathrm{r}} = 188\,\mathrm{W}$.

(e) The power of heat that the human exchanges with their surroundings by heat conduction is $P_{\mathrm{t}} = \alpha S_c (T_o - T_c) = 137\,\mathrm{W}$.

(f) To heat the room to the same temperature as the heater, the power that people exchange with their surroundings should be the same as the power of the heater. Consequently, $P_{\mathrm{g}} = N\left(P_{\mathrm{r}} + P_{\mathrm{t}}\right)$, where N is the number of persons in question. Using the solutions of parts (d) and (e), we find $N \approx 4$.

(g) The power of heat that a human exchanges with their surroundings by heat conduction is now $P_{\mathrm{t}}' = \alpha' S_c (T_o - T_c) = 73\,\mathrm{W}$, and therefore $N = \frac{P_{\mathrm{g}}}{P_{\mathrm{r}} + P_{\mathrm{t}}'} \approx 5$.

We refer the interested reader to reference [16]. Part of this problem was given by the authors at the national physics competition for the fourth grade of high school in Serbia in 2016.

Problem 4 Human Walk

How fast can humans walk without running?
What is the energy required for walking?

We consider the human walk in this problem. You might have noticed that it is quite difficult to walk fast without actually running.

(a) Make an order of magnitude estimate of the highest possible speed of the human walk without using the data given in the rest of the problem.

In the rest of this problem we consider the following model of the human walk and use it for a more detailed analysis of the human walk. We assume that the human body consists of three rigid rods that represent the two legs and the abdomen. The pelvis is where the legs and abdomen merge. The position of the abdomen during the walk is always vertical. One leg (which we call the standing leg) is always in contact with the ground, while the foot of the other leg is slightly above the ground but not in contact with the ground. Static friction between the legs and the ground is enough to prevent the standing leg from slipping. Unless otherwise stated, assume that muscles do not perform work and that the person moves only under the influence of external forces.

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Different positions that a person takes during one step are presented in Figure 1.12. At the beginning of the step (moment 1) the angle between each leg and the vertical is $\theta = \theta_0$ ($\theta_0 < 48°$). The pelvis rotates around the point of contact of the standing leg and the ground and consequently the angle θ decreases until the legs become parallel (moment 3). Next, the angle θ increases to the value $\theta = \theta_0$ (moment 5). At moment 5 there is a change of standing leg as follows. First, the standing leg loses contact with the ground (leg D in Figure 1.12 [moment 5]). Immediately after that the second leg (leg L in Figure 1.12 [moment 5′]) becomes the standing leg.

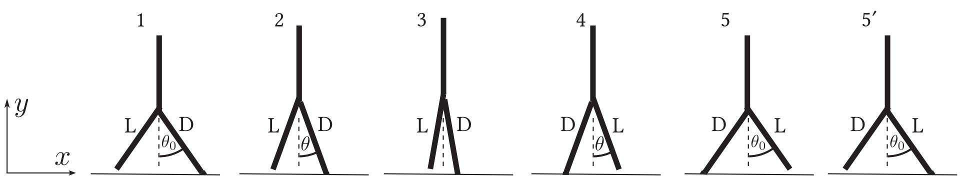

Figure 1.12 Positions a person takes during one step in five different moments of time (denoted as 1 to 5 and 5′). L denotes the left leg while D denotes the right leg

We assume for simplicity that the person's total mass is located in the pelvis. The length of each leg is l, the person's mass is m, and gravitational acceleration is g. The speed of the person at the moment when the legs are parallel is v_0.

(b) Find the expressions for the magnitude of angular velocity of the leg and for the reaction force of the ground at the moment when the angle between the leg and the vertical is θ.

(c) Determine the condition that the quantities v_0, g, l, and θ_0 should satisfy to make sure that the standing leg remains in contact with the ground.

(d) Calculate the maximal speed of walking for a person with short steps (small value of θ_0). The length of the leg is $l = 90\,\mathrm{cm}$ while gravitational acceleration is $g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$. Compare the result with the average speed of the best athlete competing in a 10 km walking race. The world record in this discipline is 37 minutes and 11 seconds for male athletes and 41 minutes and 4 seconds for female athletes. Comment on the difference in the results.

(e) The angular velocity of the leg suddenly changes during the change of the standing leg. Find the ratio of the angular velocity of the leg immediately after and immediately before the change of standing leg.

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(f) Determine the expression for the loss of kinetic energy of the person during the change of the standing leg.

(g) Calculate the work per distance traveled that muscles should perform in order to compensate for the loss of kinetic energy described in part (f). Use the following numerical values: $\theta_0 = 10^\circ$, $v_0 = 2.0\,\frac{\mathrm{m}}{\mathrm{s}}$, and $m = 75\,\mathrm{kg}$, while g and l are given in part (d).

(h) How much spinach should the person eat to have enough energy to walk the distance of $s = 10\,\mathrm{km}$? The energy value of 100 g of spinach is 28 kcal, where 1 kcal = 4.196 kJ. Use the numerical values from parts (d) and (g).

Solution of Problem 4

(a) Assume that the person walks at a speed v. For an order of magnitude estimate, we assume that the person's whole mass m is located in the pelvis. During one step the pelvis performs the motion on a circle of radius l equal to the length of the leg. The person stays in contact with the ground if the force of gravity is larger than the centripetal force $mg > \frac{mv^2}{l}$, where g is gravitational acceleration. We therefore obtain $v > \sqrt{gl}$. Assuming that $g \approx 10\,\frac{\mathrm{m}}{\mathrm{s}^2}$ and that the length of the leg is $l \approx 1\,\mathrm{m}$, we obtain a maximal speed of human walking of $v_{\max} = \sqrt{gl} \sim 3\,\frac{\mathrm{m}}{\mathrm{s}}$.

(b) The trajectory of the pelvis is the circle of radius l, which implies that its speed is $v = l\omega$, where ω is the magnitude of the angular velocity of the leg. Since we assume in this problem that the person's whole mass is located in the pelvis, the person's kinetic energy is $T(\theta) = \frac{1}{2}mv(\theta)^2 = \frac{1}{2}ml^2\omega(\theta)^2$. The gravitational potential energy (with the reference level set at the ground) is $U(\theta) = mgl\cos\theta$. The law of energy conservation reads $U(\theta)+T(\theta) = U(0)+T(0)$ and consequently

$$\omega(\theta)^2 = \frac{v_0^2}{l^2} + \frac{2g}{l}(1-\cos\theta). \tag{1.24}$$

Newton's second law of motion of the person in the y direction gives

$$\frac{\mathrm{d}p_y}{\mathrm{d}t} = N - mg, \tag{1.25}$$

where N is the magnitude of the reaction force of the ground and p_y is the y component of the person's momentum. The y coordinate of the person is given as $y = l\cos\theta$, which leads to $p_y = m\frac{\mathrm{d}y}{\mathrm{d}t} = -ml\dot{\theta}\sin\theta$, where we introduce the notation $\dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$. Further differentiation leads to

$$\frac{\mathrm{d}p_y}{\mathrm{d}t} = -ml\left(\dot{\theta}^2\cos\theta + \sin\theta\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t}\right). \tag{1.26}$$

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We further obtain

$$\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}\theta} = \frac{1}{2}\frac{\mathrm{d}\dot{\theta}^2}{\mathrm{d}\theta}. \tag{1.27}$$

Since the relation $\omega(\theta)^2 = \dot{\theta}^2$ holds at every moment of time, by using the trigonometric identity $\sin^2\theta = 1 - \cos^2\theta$, we obtain from equations (1.24) to (1.27) that

$$N = 3mg\cos^2\theta - 2mg\cos\theta - m\frac{v_0^2}{l}\cos\theta. \tag{1.28}$$

(c) The standing leg will remain in contact with the ground if the condition $N > 0$ is satisfied at every moment of time. This leads to the condition $v_0^2 < gl\,(3\cos\theta - 2)$. This condition is satisfied for each value of θ during the motion if the condition $v_0^2 < gl\,(3\cos\theta_0 - 2)$ is satisfied.

(d) Since a walk with short steps implies $\theta_0 \approx 0$, the solution of part (c) yields $v_0^{\max} = \sqrt{gl} = 3.0\,\frac{\mathrm{m}}{\mathrm{s}}$. The average speed of world record holders in walking is $v_{\mathrm{sr}} = \frac{s}{t}$, where $s = 10\,\mathrm{km}$ and $t = 2{,}231\,\mathrm{s}$ for males, while $t = 24{,}64\,\mathrm{s}$ for females. Consequently $v_{\mathrm{sr}} = 4.5\,\frac{\mathrm{m}}{\mathrm{s}}$ for males and $v_{\mathrm{sr}} = 4.1\,\frac{\mathrm{m}}{\mathrm{s}}$ for females. This result implies that the average speed of world record holders is faster than this model predicts. One reason for that could be the simplicity of the model used, which gives only an estimate of maximal possible walking speed. Another reason is that athletes perform characteristic moves of the pelvis that enable walking at higher speeds while keeping the standing leg on the ground. This is particularly important in walking events because the athlete gets a warning every time the leg loses contact with the ground and three such warnings lead to disqualification.

(e) During the change of the standing leg (from moments 5 to 5′) it loses contact with the ground first, which does not lead to a change in the person's speed. When the second leg makes contact with the ground, a sudden change takes place in the person's speed and in the angular velocity of the leg. The person's angular momentum with respect to the point of contact of the standing leg with the ground (point O in Figure 1.13) is conserved during such contact. The reason for this is that gravity is the only external force acting during the contact that has torque with respect to point O. Since the duration of the contact is short and the force of gravity is finite, the change of angular momentum with respect to point O is negligible.

Angular momentum with respect to point O before the contact of the leg with the ground is $\vec{L}_i = m\vec{r} \times \vec{v}_i$, where $\vec{r} = -l\sin\theta_0\vec{e}_x + l\cos\theta_0\vec{e}_y$ is the position of the pelvis with respect to point O; $\vec{v}_i = l\omega_i\cos\theta_0\vec{e}_x - l\omega_i\sin\theta_0\vec{e}_y$, where $\vec{e}_x$ and $\vec{e}_y$ are the unit vectors in the x and y directions; $\vec{v}_i$ is the velocity of the pelvis just before the contact of the leg with the ground (Figure 1.13);

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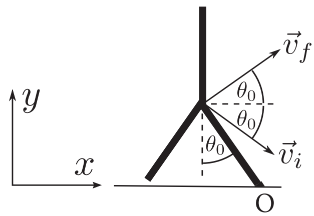

Figure 1.13 With solution of problem 4(e)

and ω_i is the magnitude of the angular velocity of the leg just before its contact with the ground. Using the identity $\cos(2\theta_0) = \cos^2\theta_0 - \sin^2\theta_0$, it follows that $(L_i)_z = -ml^2\omega_i\cos(2\theta_0)$. Next we have $\vec{L}_f = m\vec{r}\times\vec{v}_f$, where $\vec{v}_f = l\omega_f\cos\theta_0\vec{e}_x + l\omega_f\sin\theta_0\vec{e}_y$, with $\vec{v}_f$ being the velocity of the pelvis just after the contact of the leg with the ground (Figure 1.13), and ω_f being the magnitude of the angular velocity of the leg just after its contact with the ground. Using the identity $1 = \cos^2\theta_0 + \sin^2\theta_0$, it follows that $\left(L_f\right)_z = -ml^2\omega_f$. Conservation of angular momentum yields $\left(L_f\right)_z = (L_i)_z$. This leads to:

$$\frac{\omega_f}{\omega_i} = \cos(2\theta_0). \tag{1.29}$$

(f) The person's kinetic energy just before the change of standing leg is $T_i = \frac{1}{2}ml^2\omega_i^2$, while just after the change of the standing leg it is $T_f = \frac{1}{2}ml^2\omega_f^2$. Using equation (1.29) the change of kinetic energy is $\Delta T = \frac{1}{2}ml^2\sin^2(2\theta_0)\,\omega_i^2$. Using equation (1.24) we obtain

$$\Delta T = \frac{1}{2}ml^2\sin^2(2\theta_0)\left[\frac{v_0^2}{l^2} + \frac{2g}{l}(1-\cos\theta_0)\right]. \tag{1.30}$$

(g) In every step the person travels a distance of $2l\sin\theta_0$ and loses kinetic energy ΔT. The work per distance traveled that compensates for this loss of energy is $A' = \frac{\Delta T}{2l\sin\theta_0} = 59.9\,\frac{\mathrm{J}}{\mathrm{m}}$.

(h) To travel the distance of $s = 10\,\mathrm{km}$ the person requires $E = A's = 599\,\mathrm{kJ} = 143\,\mathrm{kcal}$ of energy. The energy value of spinach per unit of mass is $w = \frac{28\,\mathrm{kcal}}{100\,\mathrm{g}} = 280\,\frac{\mathrm{kcal}}{\mathrm{kg}}$. Therefore, the person has to eat $m_s = \frac{E}{w} = 0.51\,\mathrm{kg}$ of spinach.

We refer the reader interested in more detail about the physics of human walking to reference [1] and other references therein. The authors presented a modified version of this problem at the Serbian Physics Olympiad for high school students in 2018.