ON DECOMPOSABLE VARIETIES OF GROUPS

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1. Introduction

We shall be concerned with problem 7 of Hanna Neumann [7], which asks:

Prove or disprove that if $\mathfrak{U}$ and $\mathfrak{V}$ are varieties, and neither of $\mathfrak{U}$ and $\mathfrak{V}$ is contained in the other, then $\mathfrak{U} \cup \mathfrak{V}$ and $[\mathfrak{U}, \mathfrak{V}]$ are decomposable if and only if $\mathfrak{U}$ and $\mathfrak{V}$ have a common non-trivial right hand factor.

We give a negative answer to both parts of this problem. In § 2 we give a simple example which shows that $\mathfrak{U} \cup \mathfrak{V}$ can easily be decomposable even if neither $\mathfrak{U}$ nor $\mathfrak{V}$ is decomposable.

In § 4, a negative answer is given to the second part of the problem. It depends on a negative answer (in § 3) to the following problem, raised by M. J. Dunwoody in [3] in connection with Hanna Neumann's problem:

If $\mathfrak{U} > \mathfrak{U}_0$, is it true that $[\mathfrak{U}, \mathfrak{V}] > [\mathfrak{U}_0, \mathfrak{V}]$ for an arbitrary variety $\mathfrak{V}$?

Some positive partial results for the second part of Hanna Neumann's problem have been obtained by the author [2] and N. Brumberg [1].

We follow the notation of Hanna Neumann [7] and assume familiarity with the results to be found in [7].

2. A simple example

Put $\mathfrak{U} = \mathfrak{A}_5\mathfrak{A}_2 \cup \mathfrak{A}_3$, $\mathfrak{V} = \mathfrak{A}_3\mathfrak{A}_2 \cup \mathfrak{A}_5$. Then $\mathfrak{U}$ and $\mathfrak{V}$ are indecomposable by [7] 24.33. However

$$\begin{aligned} \mathfrak{U} \cup \mathfrak{V} &= (\mathfrak{A}_5\mathfrak{A}_2 \cup \mathfrak{A}_3) \cup (\mathfrak{A}_3\mathfrak{A}_2 \cup \mathfrak{A}_5) \\ &= \mathfrak{A}_5\mathfrak{A}_2 \cup \mathfrak{A}_3\mathfrak{A}_2 \\ &= \mathfrak{A}_{15}\mathfrak{A}_2, \end{aligned}$$

by [7] 21.23.

3. Dunwoody's problem

We give a negative answer to Dunwoody's problem by proving

LEMMA 1. Let $\mathfrak{U} = \operatorname{var} SL(2, 5)$, $\mathfrak{U}_0$ be the variety generated by the proper factors of $SL(2, 5)$, and $\mathfrak{V}$ be any locally finite variety of exponent prime to 30. Then $\mathfrak{U} > \mathfrak{U}_0$, and $[\mathfrak{U}, \mathfrak{V}] = [\mathfrak{U}_0, \mathfrak{V}]$.

PROOF. It follows from an examination of the proper factors of $SL(2, 5)$ and Theorem 1.5 of L. G. Kovács and M. F. Newman [5] that $SL(2, 5)$ is critical, and so $SL(2, 5) \notin \mathfrak{U}_0$. Hence $\mathfrak{U}_0 < \mathfrak{U}$.

Clearly $[\mathfrak{U}_0, \mathfrak{V}] \leqq [\mathfrak{U}, \mathfrak{V}]$.

In the other direction, first observe that $[\mathfrak{U}, \mathfrak{V}] \leqq \mathfrak{A}(\mathfrak{U} \cup \mathfrak{V})$, and so finitely generated groups in $[\mathfrak{U}, \mathfrak{V}]$ are abelian-by-finite, and hence residually finite. Thus $[\mathfrak{U}, \mathfrak{V}]$ is generated by its finite groups, and so by its critical groups.

Let G be a critical group in $[\mathfrak{U}, \mathfrak{V}]$. Since $\mathfrak{U} \cap \mathfrak{V} = E$, we have $G = U(G)V(G)$, and since $U(G) \cap V(G) = M$ centralises both $U(G)$ and $V(G)$ it follows that $M \leqq Z(G)$, the centre of G. Thus G is a central product of $U(G)$ and $V(G)$, and we may conclude from Theorem 2.1 of P. M. Weichsel [9] that G is not critical unless $G = U(G)$ or $G = V(G)$.

If $G = U(G)$, then $G \in [\mathfrak{E}, \mathfrak{V}] \leqq [\mathfrak{U}_0, \mathfrak{V}]$. Hence we may suppose that $G = V(G)$. If G is soluble, then $G/U(G) \in \mathfrak{U}_0$ and so $G \in [\mathfrak{U}_0, \mathfrak{E}] \leqq [\mathfrak{U}_0, \mathfrak{V}]$.

Hence suppose that G is insoluble. Then $G/Z_2(G) \in \operatorname{var} PSL(2, 5)$, and $G/Z_2(G)$ is insoluble, where $Z_2(G)/Z(G) = Z(G/Z(G))$. We then have that $G/Z_2(G) = H/Z_2(G) \times K/Z_2(G)$, where $H/Z_2(G)$ is isomorphic to $PSL(2, 5)$, by Lemma 3.2 of Sheila Oates [8]. Let H_0 be a minimal insoluble subgroup of G contained in H: then $H_0 = H_0'$, and $H_0/Z_2(H_0) \cong PSL(2, 5)$. From Grün's Lemma (Kurosh [6] p. 227), it follows that $Z_2(H_0) = Z(H_0)$. Since G is critical and has nontrivial centre, we get $Z(H_0) \neq 1$, and so, by V Satz 25.7 of Huppert [4], we have $H_0 \cong PSL(2, 5)$. Clearly $H_0 \cap K < M$, and so G is a central product of H_0 and K, and again by Theorem 2.1 of P. M. Weichsel [9], G is not critical unless $H_0 = G$. Thus $G \in [\mathfrak{U}_0, \mathfrak{E}] \leqq [\mathfrak{U}_0, \mathfrak{V}]$, completing the proof of the lemma.

4. The second part of Hanna Neumann's problem

Put $\mathfrak{U}_1 = \operatorname{var} SL(2, 5)$, $\mathfrak{U}_0$ the variety generated by the proper factors of $SL(2, 5)$, $\mathfrak{U} = \mathfrak{U}_0\mathfrak{A}_{11} \cup \mathfrak{U}_1$. Then we have, using Lemma 1 and [7] 21.23,

$$\begin{aligned}
[\mathfrak{U}_0, \mathfrak{A}_7]\mathfrak{A}_{11} &= [\mathfrak{U}_0\mathfrak{A}_{11}, \mathfrak{A}_7\mathfrak{A}_{11}] \\
&\leqq [\mathfrak{U}, \mathfrak{A}_7\mathfrak{A}_{11}] \\
&\leqq [\mathfrak{U}_1\mathfrak{A}_{11}, \mathfrak{A}_7\mathfrak{A}_{11}] \\
&= [\mathfrak{U}_1, \mathfrak{A}_7]\mathfrak{A}_{11} \\
&= [\mathfrak{U}_0, \mathfrak{A}_7]\mathfrak{A}_{11}.
\end{aligned}$$

Thus $[\mathfrak{U}, \mathfrak{A}_7\mathfrak{A}_{11}] = [\mathfrak{U}_0, \mathfrak{A}_7]\mathfrak{A}_{11}$. To give a negative answer to the second part of Hanna Neumann's problem, we need only show that $\mathfrak{U}$ is indecomposable. This follows from [7] 24, 33, for $\mathfrak{U} \leqq [\mathfrak{U}_0\mathfrak{A}_{11}, \mathfrak{E}]$, and $\mathfrak{U} \nleqq \mathfrak{U}_0\mathfrak{A}_{11}$, $\mathfrak{U} \nleqq \mathfrak{E}$.

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