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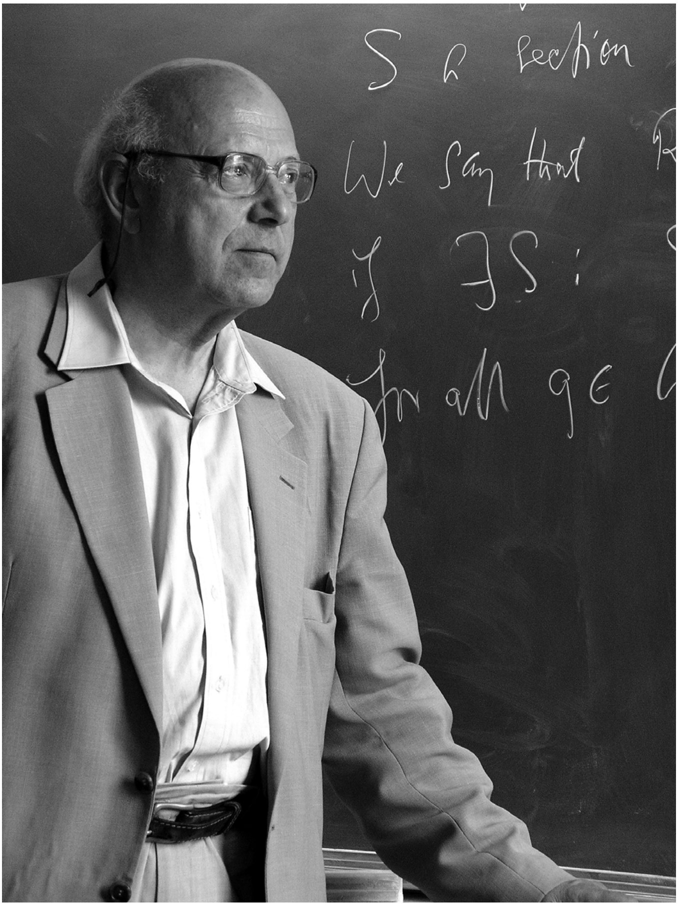


Peter Neumann
MA President 2015–2016

Inspiring Teachers

PETER M. NEUMANN

1. *Introduction*

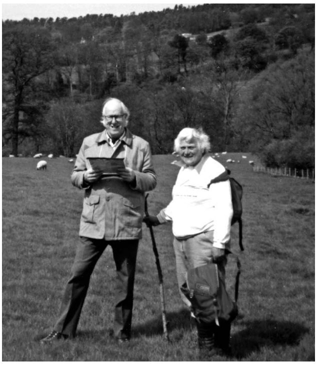
In May 2014, soon after the MA Conference in Nottingham, and soon after the passing of Roger Wheeler, a staunch member of the MA, who had been my teacher at school, I had the idea that my presidential address in April 2016 might be entitled 'Inspiring Teachers'. The concept was a talk that might trace my mathematical experiences from grammar school sixth form, through my development in retirement as a contributor to masterclasses for the UK Mathematics Trust and the Royal Institution, learning from the students and from the inspiring teachers at whose masterclasses I assist, to the small understanding of Key Stage 2 Mathematics that I am gradually acquiring through an hour a week with some Year 6 students and their remarkable teachers in a local primary school.

A month or so later, when the organising committee was giving shape to Conference 2016, my words were taken as the title of the conference itself. It is deliberately ambiguous and it was gratifying to me to find this last April that its ambiguity had struck melodious chords with other speakers. My presidential address, however, was conceived as a lecture. Like all my lectures it was designed to be an oral presentation. It was not designed to be written down and published as an article. Please bear that in mind gentle reader, and judge accordingly. If you find something of value here I shall be delighted; if not, I shall not be surprised.

2. *An inspiring teacher: Roger Wheeler* (1929–2014)

In 1961 Wheeler moved to a University Lectureship at Leicester, a post he held until his retirement in 1985. In his first few years there he

Roger Wheeler in 2008
(picture: John Watters)



Roger and Dab Wheeler in 1991
(picture: Mary Walmsley)

completed a doctorate in mathematical logic, supervised by his head of department, R. L. Goodstein (MA President 1975). John Watters, in his obituary for the *LMS Newsletter* (January 2015), writes of Wheeler's book *Rethinking Mathematical Concepts* (Ellis-Horwood 1981) ‘In this book he sought to encourage a critical attitude on the part of the reader’, a sentence that precisely captures his genius as a teacher.

After a visit to Egypt, he and Dab became expert croquet players. On retirement in 1985 they moved to the Cotswolds, to be near the headquarters of the All England Croquet Club in Cheltenham, where they contributed both as players and as organisers. The two of them also made many contributions to the MA, first through the Yorkshire branch, then at Leicester HQ, and in later life to the Gloucestershire branch. After his death on 29 March 2014, many of Roger's books came to the MA library in Leicester.

At Easter 1958 my father challenged me to factorise 99999 00000 99999 00001, a twenty-digit number. As I remember it (but my memory of detail is unreliable), he told me that one of his colleagues in the mathematics department of the University of Manchester (although he worked in Manchester, we lived in Hull) had returned from a visit to Moscow claiming that this had been set as a challenge to Russian schoolchildren. The problem took over my mind and I have always believed that it was the main reason that I failed one of my A-level GCE mathematics papers in June 1958, failed to win a state scholarship, and had to repeat the examinations in 1959. It may, however, have contributed something towards my winning a Hastings Scholarship to The Queen's College, Oxford in the Autumn of 1958. Who can know?

Where could a schoolboy start? Let

$$N = 99999\,000009\,99999\,00001.$$

It does not take long to discover that none of the prime numbers below 100 divides N. It also did not take long to notice that

$$N = \frac{10^{25} + 1}{10^5 + 1}.$$

This has the significant consequence that if p is a prime number which divides N then p must be of the form $50k + 1$. The labour of trial division was cut down by a factor of 50. Only primes such as 101, 151, … needed to be checked, and it soon emerged that $N = 251 \times 398\,40239\,04780\,87251$.

The fact that prime numbers p dividing N must have the property that $p - 1$ is a multiple of 50 was discovered by thinking about repeating decimals. Let a be an integer in the range $1 \leqslant a \leqslant p - 1$. We know that if a/p is expressed as a decimal it will have the form $0{\cdot}d_1 d_2 \ldots\ \dot{d}_k d_{k+1} \ldots\ \dot{d}_m$, ultimately repeating. This decimal expression may be calculated using long division:

$$\begin{array}{r|l} \multicolumn{1}{r}{} & 0 \cdot d_1 \;\, d_2 \;\, d_3 \ldots \\ \cline{2-2} p & a \cdot 0^{r_1}\, 0^{r_2}\, 0^{r_3} \ldots \end{array}.$$

In this notation, d_1 is the number of times that p goes into $10a$, and r_1 is the remainder, so that $10a = d_1 p + r_1$. Thus (by definition) $0 \leqslant r_1 \leqslant p - 1$, and since $a < p$, $d_1 < 10$, that is, $0 \leqslant d_1 \leqslant 9$. The process continues and at the ith stage we get $10r_i = d_{i+1}p + r_{i+1}$ with $0 \leqslant r_{i+1} \leqslant p - 1$ and $0 \leqslant d_{i+1} \leqslant 9$. Since $p > 5$ the remainders r_{i+1} will never be 0. There are $p - 1$ possible remainders, so after p steps there must be a repetition somewhere. If $r_k = r_m$, where $k < m$, and k, m are minimal then the decimal form of a/p has the recurring form described above.

Let's analyse the phenomenon that $r_k = r_m$, where $k < m$, and k, m are minimal. It means that $a \times 10^k$ and $a \times 10^m$ leave the same remainder on division by p, so p divides their difference, that is, p divides $a \times 10^k \times \left(10^{m-k} - 1\right)$. Since p is a prime number that divides neither a (because we took a in the range $1 \leqslant a \leqslant p - 1$) nor 10 (because $p > 5$), it must be the case that p divides $10^n - 1$ where $n = m - k$. Moreover, since k, m were minimal n is minimal such that p divides $10^n - 1$. It follows that $k = 0$ so that the decimal expansion of a/p is purely recurrent. Also, since n does not depend on a, and every number r in the range $1 \leqslant a \leqslant p - 1$ occurs as a digit in precisely n of the calculations producing the recurring decimal $0{\cdot}\dot{d}_1 \ldots \dot{d}_n$ (whenever it occurs, it occurs in a/p for $a = d_i$ for each i in the range $1 \leqslant i \leqslant n$), it follows that n divides $p - 1$.

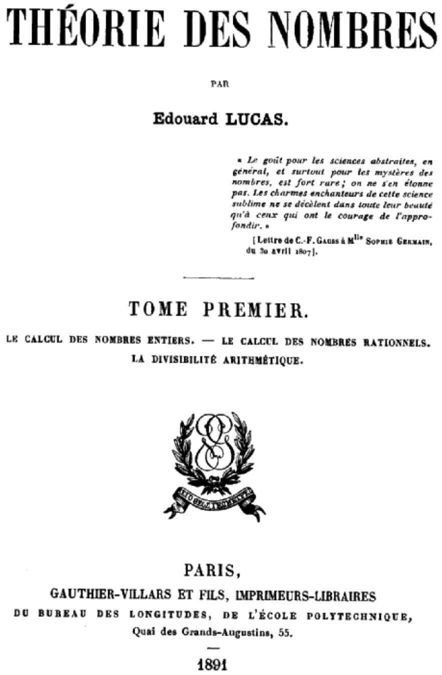
Now let's return to the case in hand. Our prime number p divides N and N divides $10^{50} - 1$. But p does not divide $10^{25} - 1$ or $10^{10} - 1$ since it divides $\left(10^{25} + 1\right)/\left(10^5 + 1\right)$ and so we infer that the period n of the decimal expansion of a/p must be 50. Therefore 50 divides $p - 1$, that is, p is of the form $50k + 1$.

It was at this point, when he saw what I was doing, that Roger Wheeler, my teacher at Hymers College, lent me one of his books, *Théorie des Nombres* (Tome Premier—but there never was a second volume) by Edouard Lucas, Paris 1891.

I learned much from Roger and this book. First, what a huge and exciting subject number theory is. Secondly, that the fact that a prime divisor p of N must be of the form $50k + 1$, and the reasons behind this, are part of a well-known theory surrounding lovely facts known as Fermat's little theorem and Lagrange's theorem in group theory. Fermat's little theorem, that if p is prime and a is any integer then $a^p - a$ is divisible by p (or its equivalent version, that if p is prime and a is any integer not divisible by p then $a^p - 1$ is divisible by p), has much nicer proofs than the one I have sketched above. Third, I learned that Lucas, after whom the sequence 1, 3, 4, 7, 11, 18, … of 'Fibonacci type', is named, had been much concerned with factorising numbers of the form $a^n - 1$. I am not certain, but some evidence suggests that it was Lucas himself, using quite deep primality tests and factorisation methods based on properties of his eponymous sequence, who found, some 140 years ago that

$$\begin{aligned} 99999\,00000\,99999\,00001 &= 251 \times 398\,40239\,04780\,87251 \\ &= 251 \times 5051 \times 7887\,59434\,72201, \end{aligned}$$

and that all those last three numbers are prime.



THÉORIE DES NOMBRES

PAR

Edouard LUCAS.

« *Le goût pour les sciences abstraites, en général, et surtout pour les mystères des nombres, est fort rare; on ne s'en étonne pas. Les charmes enchanteurs de cette science sublime ne se décèlent dans toute leur beauté qu'à ceux qui ont le courage de l'approfondir.* »

[Lettre de C.-F. GAUSS à M$^{\text{lle}}$ SOPHIE GERMAIN, du 30 avril 1807].

TOME PREMIER.

LE CALCUL DES NOMBRES ENTIERS. — LE CALCUL DES NOMBRES RATIONNELS.
LA DIVISIBILITÉ ARITHMÉTIQUE.

PARIS,
GAUTHIER-VILLARS ET FILS, IMPRIMEURS-LIBRAIRES
DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,
Quai des Grands-Augustins, 55.
—
1891

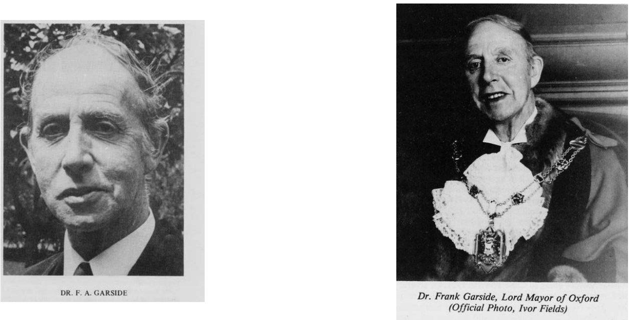
This picture shows the title page and the motto translates as:

> A taste for abstract sciences in general, and especially for the mysteries of numbers, is very rare; that is not surprising. The enchanting charms of this sublime science do not reveal themselves in all their beauty except to those who have the courage to deepen it.
>
> [Letter of 30 April 1897 from C. F. Gauss to M$^{\text{lle}}$ Sophie Germain]

Factorising big numbers: how exactly did Edouard Lucas accomplish this in about 1878? How far could a schoolboy get in 1958? How quickly did a little hand-held machine do it in 1978? And now? Nowadays, since about 1970, factorisation is an area of practical concern because of its application in cryptography and for that reason, there has been much research combining number theory and computer science. A 20-digit number can now be factorised in a microsecond or less on an ordinary laptop machine; even 100-digit numbers take only a small fraction of a second; a 300-digit number can still be something of a challenge, though, and a 600-digit number that is a product of two different 300-digit prime numbers is still well-nigh impossible in a lifetime.

3. *An inspiring teacher and mathematician: F. A. Garside*

Frank Arnold Garside was born on 27 April 1915 in Salford (now Greater Manchester). He was educated at The Manchester Grammar School and Corpus Christi College, Oxford, where he achieved First-class Honours in Mathematics in June 1936 and Third-class honours in Geography one year later. He taught for two years in Ramsgate, then enlisted in the army and served for the duration of World War II in India and the Middle East, rising to the rank of Major.



DR. F. A. GARSIDE

Dr. Frank Garside, Lord Mayor of Oxford
(Official Photo, Ivor Fields)

Frank Garside (1915–1988): teacher, mathematician, local politician

After demobilisation, in January 1947 he returned to Oxford where he taught mathematics at Magdalen College School until he retired in July 1978. He was a staunch contributor to the activities of the school CCF (Combined Cadet Force), in which he was always known as Major Garside. According to A. N. Middleton, to whom I am greatly indebted for information about his teaching career, he was known to students as 'FAG', only partly in reference to his initials. He smoked incessantly though (as I remember him) in a naturally casual and leisurely sort of way. He served as Head of Department for his last two years, 1976–1978.

Like Roger Wheeler, Frank Garside was active in the MA. He was one of the founders of the ODBMA, the Oxford and District Branch of the Mathematical Association, and served as its secretary during the 1960s. That was when, as student representative on its committee, I came to know and admire him. He represented the ODBMA on MA Council 1961, and served again on Council 1966–1971 as Honorary Assistant Secretary, a title that no longer exists.

At this time he was also active politically. He was elected as a Conservative to Oxford City Council in 1967. He lost his seat in 1971, but regained it the next year, and remained a Councillor for another sixteen years until the end of his life in 1988. He served as Sheriff of Oxford 1980–1981 and as Lord Mayor 1984–1985.

Frank carried his enthusiasm for mathematics rather further than is usual for someone with a full-time teaching load. In 1958 he embarked on an Oxford doctorate. His subject was the algebra (group theory) associated with knots and braids. He was awarded an Oxford DPhil in 1965 for his thesis *The theory of knots and associated problems*. Out of that came a remarkable article, 'The braid group and other groups' published in the *Quarterly Journal of Mathematics* (Oxford) in 1969. According to *MathSciNet* (*Mathematical Reviews* online), in March 2016 it had been cited 203 times (for comparison, my own most-cited article had been cited 42 times); the word 'Garside', now an eponymous technical term, occurred in titles of 57 articles and books on topics described as 'Garside theory', 'Garside groups', 'Garside systems' and many, many more. Early in the 1980s he resumed his study of braid groups, seeking to use a primitive little Sinclair computer to settle an old conjecture relating them to matrices. Once in a while he would send me a short note or question about them, written on headed notepaper from the Oxford Lord Mayor's Parlour, but he made no further useful progress. Nevertheless, his 1959 article had won him admirers all over the world—a fact that he never quite understood, though he was always very kind, helpful and polite. He appreciated it, however, when a distinguished professor of mathematics from Illinois chose to spend a sabbatical year in Oxford in order (he told me) that his sons might attend Magdalen College School to be taught by the great Dr Garside.

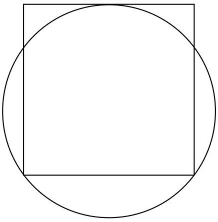
4. *What inspires? A Y12 episode*

If you are still with me, gentle reader, then pause a moment, as we did in the lecture, to ask the question:

What piece(s) of mathematics inspired YOU?

In my case, over my long professional life there have been many pieces of mathematics that have excited me. Most have been associated with research or with teaching at undergraduate or postgraduate level, but in the last few years a good number have come from working with schoolchildren.

Here is something that caught my eye while I was actively enjoying the presidency of the MA. A circle touches one side of a square and passes through the end points of the opposite side. *Which is longer—the perimeter of the square or the circumference of the circle?*



Which is longer?

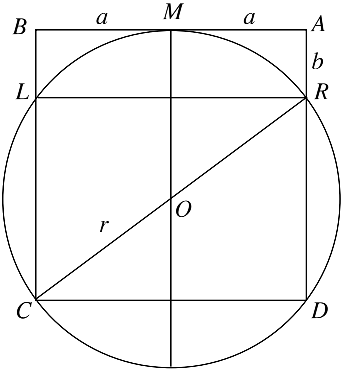
I owe this to Jennie Golding, the current President, who offered it in one of her sessions at the MA Secondary Teachers' Conference, Stirling, September 2015.

I found this to be a good question for discussion with a group of about twenty Year 12 students. Circumstances were not ideal. We had been given a raked lecture theatre with fixed seating, not really suitable for what was intended as a kind of tutorial session. Nevertheless, it was not hard to get the students to talk about it in small groups before we moved to general discussion.

- "It could be useful to join that point on the right to the one on the left";
- "Do you mean this one to that one?";
- "No sir, the ones where the circle touches the square";
- "The point where they touch is on the top, not on the right. Ah! do you mean these? I don't think they really touch there, do they; I'd say they cross."

It soon emerged that exchanges like these were a little cumbersome, and a couple of girls broke in to suggest that letters—names for the points of interest—would help. That was a turning point. Such a simple idea, but with labels it all became so much easier not only to talk about the mathematics but also to see how one might proceed!

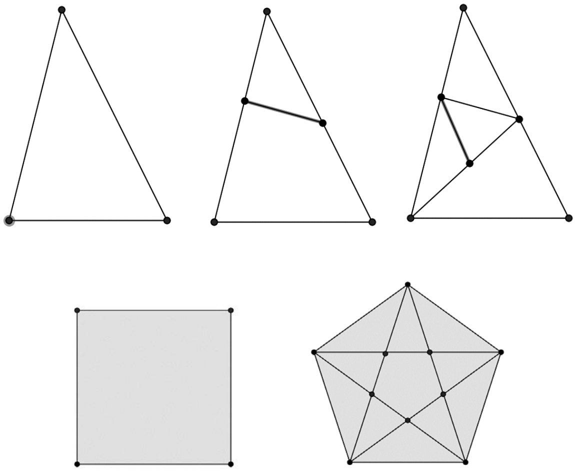
The square could be *ABCD*, the point where circle and side of square touched could be *M* because the students had realised quite quickly that it was the mid-point of the 'top' side *AB*. The points where the circle met the 'right' and 'left' sides could be *R* and *L*, the centre of the circle might be *O*. Then there were suggestions as to construction lines that might be added, such as *RL* and the diameter of the circle that passes through *M*, quickly spotted to be the perpendicular bisector of *AB* and also of *RL*. Perhaps it would be helpful to have $2a$ for the side-length of the square, so that $AM = MB = a$. And b for *AR*. And r for the radius *OC* of the circle. Then one of the girls spotted that *CR* must be another diameter of the circle.



After that some ideas how to find b and r in terms of a came up, rather tentatively at first. But by the time the session had to finish a few of the students had made good progress. Sadly, I heard no more. Did they discover that $r = 5a/4$? Did they realise that if c is the circumference of the circle and s the perimeter of the square then $c/s = 5\pi/16$? Were they comfortable that $5\pi < 16$ so that the answer is $c < s$? I'd like to hope so, but as I said, I heard no more. In fact, taking 22/7 as an approximation to π (a pretty good one, accurate to about 0.04%) we find that c/s is very close to $110/112$ and so the perimeter of the square is approximately 2% greater than the circumference of the circle, a small enough difference that it is not visible to the naked eye.

5. *What inspires? Two Y6 episodes*

Four years ago, working for 45 minutes each week for four weeks with a group of five girls in the top two years of a local primary school, I drew these pictures.



And we discussed these questions:

- How many points?
- How many line pieces?
- How many areas?
- Points + areas − lines = ?

We had started with a map of the counties of England, but that was far too complicated. So over a couple of sessions we simplified, and treated much simpler configurations, of which those shown above were a small sample. The answer always came out to be the same. The language is not what is now conventional. We would usually write $V - E + F = 1$ (remember, these are planar maps, not polyhedra). But this is the language that the girls themselves developed, and it seemed as good as any at the time. The challenge was to find me a map (just one country, no islands, no inland seas) for which the equation

$$\text{points} + \text{areas} - \text{lines} = 1$$

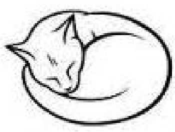
is false, or to explain why for any map (just one country, no islands, no seas) the equation

$$\text{points} + \text{areas} - \text{lines} = 1$$

will be true.

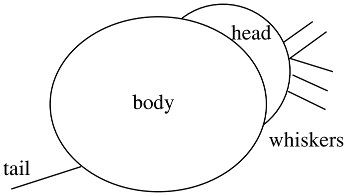
The rules—perfectly acceptable in terms of maps showing counties or countries—were that each line had two ends (so a circle had to have at least one point on it), that an area had to have a boundary going all the way round, and that every point and line was connected to the rest of the diagram. That is what we took ‘one country, no islands, no inland seas’ to mean.

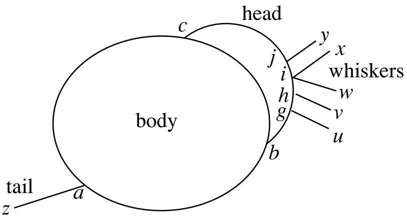
The girls tried larger and larger and more complicated examples, always hopeful. But when we counted carefully we found the same answer, the number always turned out to be 1. Then one of the girls enlisted a picture of her cat:



But even her cat had points + areas − lines = 1. She tried adding more whiskers, and more ... and more ... : still the same.

Here is the student's example made crude:





In this example there are 13 points, 14 line pieces and 2 areas, so

$$\text{points} + \text{areas} - \text{lines} = 13 + 2 - 14 = 1.$$

What happens when we add a whisker? If the whisker grew from a point that was already in the picture then it added 1 to the lines-count and 1 to the points-count (namely, the point at the other end of the whisker) so the quantity ‘points + areas − lines’ had 1 point added and 1 line taken away, and remained the same as it had been before. Otherwise, if it grew from a new point on the cat's face then it added 2 to the points count and 2 to the line count: the whisker itself and the fact that one segment of the boundary of the cat's face had been cut into two. In this case, the quantity ‘points + areas − lines’ had 2 points added and 2 lines taken away, and again it remained the same as it had been before.

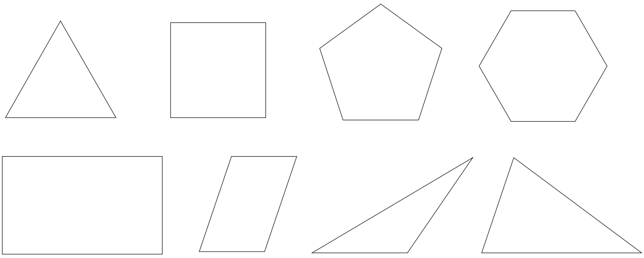
But this now began to give some insight. It was not long before the girls discovered that they could extend this argument and explain, if in a halting and unconfident way, that they could never find a map for which the equation

$$\text{points} + \text{areas} - \text{lines} = 1$$

is false; that is, this equation would always be true.

I learned a lot from this. First that children have different ways of thinking—different from each other's and happily different from mine; secondly, that there are aspects of Euler's theorem for planar maps that I had not seen before; thirdly, how very much of a privilege it is, not to be confined by a syllabus or other expectations.

Here is another example of a similar phenomenon. This was in January this year. I had four sessions of about 45 minutes with a group of nine Y6 students, four girls, five boys. They had requested to work on tessellations, because (I think) they had coloured some in an art session and their teacher had told them it was a mathematical topic.



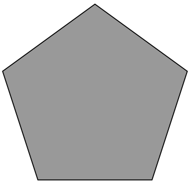
Which of these shapes can be used for tiling?

In the first few minutes we had various answers: 'the square can be used, none of the others can'; 'the square and the rectangle can, none of the others can'. Gradually conviction grew that the equilateral triangle, the square, the regular hexagon, the rectangle and the parallelogram could be used, and in each case most of the children showed tilings. That left the regular pentagon and the two other triangles. Two or three of the boys were confident that 'You can't use the pointy triangles', to which the answer had to be 'then explain to us why not'. After a bit, all the children had realised that two copies of a triangle, however 'pointy' it is, can be put together to make a parallelogram, and that a parallelogram can be used to tile in strips. Here there was a subsidiary question—if a shape can be used as a tile, in how many different ways can a tiling be arranged? But let's leave that aside.

We had arrived at the belief that any of those shapes may be used for tiling, **except** the regular pentagon.

But **why** not the pentagon?

Because its angles are 108°: at a corner four would be too big; three would be too small.



One of the girls had realised that the angles matter, had worked them out by dividing the pentagon into triangles, and had realised that four pentagons could not fit round a corner, three would not fill the corner. Although I had estimated her to be the most diffident (indeed, the only diffident child in a group where most of the others are very outgoing) I asked if she would mind giving a little lecture to explain her ideas to the other eight. She did. For me that was a very moving moment.

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