Canad. Math. Bull. Vol. 17 (3), 1974

ON ADJACENCY PRESERVING MAPS

BY
R. WESTWICK

1. In his paper [1] on homogeneous spaces W. L. Chow states that "Any one-to-one adjacency preserving transformation of the Grassmann space of all the $[r]$ of $S_n(0<r<n-1)$ onto itself is a transformation of the basic group of the space." In the proof both the transformation and its inverse are assumed to be adjacency preserving. See also Dieudonne [2] p. 81. What we show in this paper is that the inverse of a one-to-one onto adjacency preserving transformation is itself adjacency preserving and so Chow's theorem is in fact correct as stated.

2. To fix the notation we let U denote a finite dimensional vector space over a field F. We let $P_r(U)$ denote the set of all r-dimensional subspaces of U. It is convenient also to introduce, for each subspace V of U, the set $Q_r(V)$ consisting of the set of all r-dimensional subspaces of U containing V. The purpose of this note is to prove the

THEOREM *Let f: $P_r(U)\rightarrow P_r(U)$ be a one-to-one onto adjacency preserving transformation. Then f^{-1} preserves adjacency also.*

The proof depends on determining the effect of f on the maximal sets of pairwise adjacent subsets of $P_r(U)$. The two possible types of maximal sets are the $P_r(V)$ for $V\in P_{r+1}(U)$ and the $Q_r(V)$ for $V\in P_{r-1}(U)$. We let $\mathfrak{A}_r(U)=\{P_r(V): V\in P_{r+1}(U)\}$ and $\mathfrak{B}_r(U)=\{Q_r(V): V\in P_{r-1}(U)\}$. The proof separates into two parts depending on whether there are two members of $\mathfrak{A}_r(U)$ whose images are of different type or all the members of $\mathfrak{A}_r(U)$ have images of the same type. In the next section we will show that the first alternative is not possible by showing that f cannot be onto in that case.

3. We assume that there are two elements $P_r(V_1)$ and $P_r(V_2)$ of $\mathfrak{A}_r(U)$ such that $f(P_r(V_1))\subseteq A$ for some $A\in\mathfrak{A}_r(U)$ and $f(P_r(V_2))\subseteq B$ for some $B\in\mathfrak{B}_r(U)$. Then there must be an adjacent pair of subspaces V_1 and V_2 satisfying the above and so we may assume this to be the case at the outset.

In this paragraph we show that there is an $A_1\in\mathfrak{A}_r(U)$ and $B_1\in\mathfrak{B}_r(U)$ (not necessarily the A and B above) for which we have $f(P_r(V_1+V_2))\subseteq A_1\cup B_1$. For each $V\in P_{r-1}(V_1\cap V_2)$ the set $f(Q_r(V))$ has at least two points (r-dimensional subspaces of U will be referred to as points) in common with each of A and B. Therefore $f(Q_r(V))\subseteq A$ or $f(Q_r(V))\subseteq B$. For each $W\in Q_{r+1}(V_1\cap V_2)$ we have $(P_r(W))\subseteq A\cup B$ (because every r-dimensional subspace of W meets $V_1\cap V_2$ in at least $(r-1)$-dimensions) and therefore $f(P_r(W))\subseteq A$ or $f(P_r(W))\subseteq B$. Then there are at least two members of $\mathfrak{A}_r(U)$ which are mapped into one of A or B and for

https://doi.org/10.4153/CMB-1974-074-7 Published online by Cambridge University Press

definiteness we will take it to be A. Then for every $W \in P_{r+1}(V_1+V_2)$ such that $f(P_r(W))$ has the same type as A we have $f(P_r(W)) \subseteq A$ because it meets A in at least two points. If all the $f(P_r(W))$ are of the same type as A then we may take $A_1 = A$ and $B_1 = B$. Suppose then that for some $W_0 \in P_{r+1}(V_1+V_2)$, $f(P_r(W_0))$ is not of type A. Then $f(P_r(W_0)) \subseteq B_1$ for some $B_1 \in \mathfrak{B}_r(U)$ and there is a point $X \in P_r(W_0)$ such that $f(X) \notin A$. We point out here that B_1 need not be B because in our choice of V_1 and V_2 it may have happened that $f(P_r(V_2)) \subseteq B \cap A$. By considering the pair $P_r(V_1)$ and $P_r(W_0)$ we have $f(Q_r(V)) \subseteq A \cup B_1$ for all $V \in P_{r-1}(V_1 \cap W_0)$. If $V = X \cap V_1 \cap W_0$ then $f(Q(_rV)) \subseteq B_1$ and for any $W \in Q_{r+1}(X)$, $f(P_r(W))$ cannot be of type A. Therefore $f(P_r(W))$ has the same type as B_1 and since it meets B_1 in at least two points we have $f(P_r(W)) \subseteq B_1$. Therefore at least two members of $\mathfrak{A}_r(U)$ are mapped into subsets of B_1 and so we have $f(P_r(V_1+V_2)) \subseteq A \cup B_1$. Note that $A_1 \cap B_1$ is not empty since in the first case $A_1 \cap B_1 = A \cap B$ contains $f(V_1 \cap V_2)$ while in the second case $A_1 \cap B_1$ contains $f(V_1 \cap W_0)$.

To show that f is not onto we consider first the case $n \leq 2r$. Let $A_1 = P_r(W_1)$ and $B_1 = Q_r(W_2)$ where $W_1 \in P_{r+1}(U)$ and $W_2 \in P_{r-1}(U)$. Since $A_1 \cap B_1$ is not empty we have $W_2 \subseteq W_1$. Choose $X \in P_r(U)$ so that $\dim(X \cap W_1)$ and $\dim(X \cap W_2)$ are both minimal. Then any chain of points $X_1, X_2, \ldots, X_k$ of $P_r(U)$ where X_i, X_{i+1} are adjacent, $X = X_1$ and $X_k \in A_1 \cup B_1$ has length at least $k = n-r$ (If $X_k \in A_1$ or if $n = 2r$ then $k \geq n-r$ while if $X_k \in B_1$ and $n < 2r$ then $k \geq n-r+1$). For any $Y \in P_r(U)$ there is a chain $Y_1, Y_2, \ldots, Y_{n-r-1}$ of points of $P_r(U)$ such that Y_i, Y_{i+1} are adjacent, $Y = Y_1$, and $Y_{n-r-1} \in P_r(V_1+V_2)$. Therefore X cannot be the image of any Y and so f is not onto. The case for which $n > 2r$ is reduced to the previous case as follows; Let $g\colon P_r(U) \to P_{n-r}(U)$ be a map induced by a correlation and consider the map

$$f' = g \circ f \circ g^{-1}\colon P_{n-r}(U) \to P_{n-r}(U).$$

Then f' is a one-to-one onto adjacency preserving map and for each $V \in P_{r-1}(V_1 \cap V_2)$, f' maps $g(Q_r(V))$ into one of $g(A)$ or $g(B)$. Now, $g(A) \in \mathfrak{B}_{n-r}(U)$ $g(B) \in \mathfrak{A}_{n-r}(U)$, and $g(Q_r(V)) \in \mathfrak{A}_{n-r}(U)$. We select a distinct pair V' and V'' from $P_{r-1}(V_1 \cap V_2)$ and choose W_1' and W_2' from $P_{n-r+1}(U)$ such that $g(Q_r(V')) = P_{n-r}(W_1')$ and $g(Q_r(V'')) = P_{n-r}(W_2')$. Then W_1' and W_2' are adjacent. If f' takes each member of $\mathfrak{A}_{n-r}(W_1'+W_2')$ into sets of the same type then since at least two of them go into one of $g(A)$ or $g(B)$ (namely the $g(Q_r(V))$ for $V \in P_{r-1}(V_1 \cap V_2)$) they all do. If some pair has images of different type then we are back to the original hypothesis of this section and with $n < 2(n-r)$. Therefore f' is not onto and it follows that f is not onto.

4. In this section we assume that f maps the members of $\mathfrak{A}_r(U)$ into sets all of the same type, and the members of $\mathfrak{B}_r(U)$ into sets all of the same type. For any two members of $\mathfrak{A}_r(U)$ with underlying adjacent $(r+1)$-dimensional subspaces the maximal sets containing their images under f must be distinct. This follows from the argument in §3. Therefore for $A \in \mathfrak{A}_r(U)$ and $B \in \mathfrak{B}_r(U)$, the types of

https://doi.org/10.4153/CMB-1974-074-7 Published online by Cambridge University Press

$f(A)$ and $f(B)$ are different. By replacing f by f^2 if necessary, we may assume that $f(A) \subseteq A' \in \mathfrak{A}_r(U)$ for $A \in \mathfrak{A}_r(U)$ and $f(B) \subseteq B' \in \mathfrak{B}_r(U)$ for $B \in \mathfrak{B}_r(U)$.

We will assume that the theorem is true for spaces U' where $\dim U' < \dim U$. Note that when $\dim U = r+1$ the theorem is obvious because in this case every pair of points of $P_r(U)$ are adjacent.

We begin by proving the statement: If V_1 and V_2 are adjacent k-dimensional subspaces of U and if $P_r(W)$ contains both $f(P_r(V_1))$ and $f(P_r(V_2))$ then $P_r(W)$ contains $f(P_r(V_1+V_2))$. When $k=r$ this statement is certainly valid, and we proceed from here by induction. Let $V_0 \in P_r(V_1+V_2)$ with $V_0 \notin P_r(V_1)$ and $V_0 \notin P_r(V_2)$. Then $V_0 \nsubseteq V_1 \cap V_2$ and so we can select $V \in P_k(V_1+V_2)$ which contains V_0 but not $V_1 \cap V_2$. Then $V \cap V_1$ and $V \cap V_2$ are adjacent $(k-1)$-dimensional subspaces and $P_r(W)$ contains $f(P_r(V \cap V_1)$ and $f(P_r(V \cap V_2)$. Therefore $f(P_r(W))$ contains $f(P_r(V))$ and so $f(V_0) \in P_r(W)$ which completes the proof of the statement. By an induction argument it now follows that if $V \in P_k(U)$ and $V_0 \in P_r(U)$ such that $V+V_0 \in P_{k+1}(U)$ and if W is a subspace of U such that $P_r(W)$ contains $f(P_r(V))$ and $f(V_0)$ then it contains $f(P_r(V+V_0))$. Therefore for each $V \in P_k(U)$ there is a $W \in P_k(U)$ such that $f(P_r(V)) \subseteq P_r(W)$. When $k=n-1$ we must in fact have equality, for suppose $W_0 \in P_r(W)$ and $f^{-1}(W_0) \notin P_r(V)$. Then $U=V+f^{-1}(W_0)$ and so $f(P_r(U)) \subseteq P_r(W) \neq P_r(U)$, which contradicts the assumption that f is onto.

We now conclude our proof of the main theorem as follows: Let U_1 and U_2 be adjacent points and let $V_1=f^{-1}(U_1)$, $V_2=f^{-1}(U_2)$. Let W_1 be a hyperspace of U containing V_1. If W_1 contains V_2 then V_1 and V_2 are adjacent by the induction hypothesis, and so we suppose $V_2 \nsubseteq W_1$. Let W_2 be the hyperspace of U for which $f(P_r(W_1))=P_r(W_2)$. Then $U_1 \subseteq W_2$ and $U_1 \cap U_2$ is an $(r-1)$-dimensional subspace of W_2. Let $V_3 \in P_r(W_1)$ be adjacent to V_2. Then $f(V_3) \subseteq W_2$ and meets U_2 in an $(r-1)$-dimensional subspace and since $\dim(U_2 \cap W_2)=r-1$ we have $f(V_3) \cap U_2 = U_2 \cap W_2 = U_1 \cap U_2$. Therefore $f(V_3)$ is adjacent to $f(V_1)$ and so V_3 is adjacent to V_1 (both are contained in W_1). Then the images of $Q_r(V_1 \cap V_3)$ and $Q_r(V_2 \cap V_3)$ are contained in $Q_r(U_1 \cap U_2)$ from which we have $V_1 \cap V_3 = V_2 \cap V_3$. Therefore V_1 and V_2 are adjacent.

References

1. W. L. Chow, *On the Geometry of Algebraic Homogeneous Spaces*. Ann. of Math. **50** (1949), 32–67.

2. J. Dieudonné, *La Géométrie Des Groupes Classiques*. 3rd edition, Springer-Verlag, Berlin Heidelberg New York (1971).