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THE METRISABILITY OF PRECOMPACT SETS

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A uniform space is trans-separable if every uniform cover has a countable subcover. We show that a uniform space is trans-separable if it contains a suitable family of precompact sets. Applying this result to locally convex spaces, we are able to deduce that the precompact subsets of a wide class of spaces are metrisable. The proof of our main Theorem is based on a cardinality argument, and is reminiscent of the classical Bolzano–Weierstrass Theorem.

A uniform space X is said to be *trans-separable* if every uniform cover of X admits a countable subcover. Expressed in terms of entourages or surroundings, this means that for each entourage H in $X \times X$, there is a countable subset C of X satisfying $H[C] = X$. This condition is strictly weaker than the usual topological notion of separability: every separable uniform space is trans-separable, but not conversely. The class of trans-separable spaces forms a reflective subcategory of the category of uniform spaces; it is closed under the taking of completions, subspaces, products, uniformly continuous images and projective limits. Every Lindelöf uniform space is trans-separable, but again the converse does not hold.

The concept of trans-separability is sometimes encountered in the context of topological vector spaces. Pfister [6] has shown that a locally convex space is trans-separable if and only if the equicontinuous subsets of its dual are weak* metrisable. (For simplicity, we shall assume that all our spaces are Hausdorff.) As a consequence, many results in functional analysis which are normally stated in terms of separable spaces actually only require the weaker condition of trans-separability.

Trans-separability appears under many different names in the literature. While workers in the field of uniform spaces tend to speak simply of separable spaces (see [3] and [4]), those who study locally convex spaces favour the phrase "separable by seminorm". Pfister investigated topological vector spaces which were "of countable type", whereas Drewnowski [2] coined the word "trans-separable" while working with topological abelian groups.

Pfister used the idea of trans-separability to show that in a DF space precompact sets are metrisable. (For the definition of a DF space, and other terms from the theory

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of locally convex spaces, see [5].) More recently, Cascales and Orihuela [1] were able to exhibit an even larger class of spaces sharing this property. Let $\mathcal{N}$ be the set of all functions $\alpha: \mathbf{N} \to \mathbf{N}$, and for each $\alpha, \beta \in \mathcal{N}$, define $\alpha \leqslant \beta$ to mean that $\alpha(k) \leqslant \beta(k)$ for all $k \in \mathbf{N}$. Cascales and Orihuela studied the class of all locally convex spaces E whose duals contained a family $\{A_\alpha : \alpha \in \mathcal{N}\}$ of subsets satisfying the following conditions:

(1) $\bigcup_{\alpha \in \mathcal{N}} A_\alpha = E'$,

(2) $(\forall \alpha, \beta \in \mathcal{N})\ \alpha \leqslant \beta \Rightarrow A_\alpha \subseteq A_\beta$, and

(3) the countable subsets of each A_α are equicontinuous on E.

(Here E' denotes the topological dual of the locally convex space E.) It is possible to show that this class includes both the DF and the LF spaces, and that it enjoys a wide range of permanence properties. To prove that the precompact subsets of the spaces in this class are metrisable, Cascales and Orihuela made use of ideas from descriptive set theory. In this note we shall derive a similar result by more elementary means.

A link between trans-separability and the metrisability of precompact sets is provided by the concept of the *polar dual* E'_p of a locally convex space E. This is the dual of E equipped with the topology of uniform convergence on the precompact subsets of E. It follows immediately from the Grothendieck Interchange Lemma that the equicontinuous subsets of E' are precompact in E'_p.

LEMMA. *If the polar dual E'_p of a locally convex space E is trans-separable, then the precompact subsets of E are metrisable.*

PROOF: On the precompact subsets of E, the weak* topology $\sigma(E'', E')$ coincides with given topology. These subsets are equicontinuous on E'_p, so by the result of Pfister's mentioned above they are $\sigma(E'', E')$-metrisable. ∎

For each $\alpha \in \mathcal{N}$ and $n \in \mathbf{N}$, put

$$\alpha \mid n = \{\beta \in \mathcal{N} : \alpha(k) = \beta(k) \text{ for } k = 1, 2, \ldots, n\}.$$

The elements of $\mathcal{N}$ can of course be thought of as being sequences of positive integers. At the same time, it is convenient to identify $\alpha \mid n$ with the finite sequence $\alpha(1), \alpha(2), \ldots, \alpha(n)$. Note that each $\alpha \mid n$ can be written in the form

$$\alpha \mid n = \bigcup \{\beta \mid n+1 : \beta \in \alpha \mid n\}.$$

An important point is that the union on the right-hand side involves only countably many distinct sets, which can be indexed by the values of $\beta(n+1)$, where β ranges through $\alpha \mid n$.

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Theorem. *Let X be a uniform space. If there exists a family $\{P(\alpha)\colon \alpha \in \mathcal{N}\}$ of precompact subsets of X such that*

(1) $\bigcup_{\alpha \in \mathcal{N}} P(\alpha) = X$, and

(2) $(\forall \alpha, \beta \in \mathcal{N})\ \alpha \leqslant \beta \Rightarrow P(\alpha) \subseteq P(\beta)$,

then X is trans-separable.

Proof: Suppose that X is not trans-separable. Using Zorn's Lemma, we can find an entourage H in $X \times X$ and an uncountable subset A of X such that

$$(x, y) \in H \Longrightarrow x = y \text{ for all } x, y \in A.$$

No infinite subset of A can be precompact, yet we shall select a sequence of distinct points from A so that it lies in one of the $P(\alpha)$'s. This will violate the precompactness of that $P(\alpha)$.

For each $\alpha \in \mathcal{N}$ and $n \in \mathbf{N}$, put

$$P[\alpha \mid n] = \bigcup\{P(\beta) : \beta \in \alpha \mid n\}.$$

First note that $A \subseteq \bigcup\{P[\alpha \mid 1]\colon \alpha \in \mathcal{N}\}$. This union comprises only countably many distinct sets, so we can apply the Pigeonhole Principle to choose an $\alpha_1 \in \mathcal{N}$ such that $P[\alpha_1 \mid 1]$ contains uncountably many points from A. In fact, we can choose α_1 so that $P(\alpha_1)$ itself contains at least one point x_1 from A.

Suppose that for some $n \in \mathbf{N}$, we have selected $\alpha_n \in \mathcal{N}$ and $x_1, x_2, \ldots, x_n \in A$, and $P[\alpha_n \mid n]$ contains uncountably many points from A. Note that

$$P[\alpha_n \mid n] = \bigcup\{P[\beta \mid n+1] : \beta \in \alpha_n \mid n\}.$$

Again this union involves only countably many distinct sets. There must be an $\alpha_{n+1} \in \alpha_n \mid n$ such that $P[\alpha_{n+1} \mid n+1]$ contains uncountably many points from A. It is not difficult to choose this α_{n+1} so that $P(\alpha_{n+1})$ contains an x_{n+1} that belongs to A and which is different from $x_1, x_2, \ldots x_n$.

For each $k \in \mathbf{N}$, we have that $\alpha(k) = \alpha_k(k)$ for all $n > k$. Define $\gamma \in \mathcal{N}$ by

$$\gamma(k) = \max\{\alpha_1(k), \alpha_2(k), \ldots, \alpha_k(k)\} \text{ for all } k \in \mathbf{N}.$$

Note that each $\alpha_n \leqslant \gamma$. This means that $x_n \in P(\alpha_n) \subseteq P(\gamma)$ for all $n \in \mathbf{N}$, leading to the desired contradiction. ∎

The statement of the first corollary to this Theorem appears as Satz 1 in [6].

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Corollary 1. *Let E be a topological vector space. If there exists a surjective linear map T from a metrisable topological vector space F onto E, such that the image under T of each bounded sequence in F is a precompact subset of E, then E is trans-separable.*

Proof: Let $\{V_n : n \in \mathbf{N}\}$ be a countable local base for the neighbourhoods of 0 in F. For each $\alpha \in \mathcal{N}$, $\bigcap_{k=1}^{\infty} \alpha(k) V_k$ is a bounded subset of F, so we can define $P(\alpha) = \bigcap_{k=1}^{\infty} \alpha(k) T(V_k)$. ∎

This corollary is usually applied in the case when T is the identity map on a space F equipped with two locally convex topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, where $\mathcal{T}_1 \subseteq \mathcal{T}_2$, $\mathcal{T}_2$ is metrisable and the $\mathcal{T}_2$-bounded subsets of F are $\mathcal{T}_1$-precompact. For example, if a locally convex space E contains a fundamental sequence of bounded sets, then its dual E' is $\beta(E', E)$-metrisable. If in addition E is quasi-ℓ^∞-barrelled, then every $\beta(E', E)$-bounded sequence in E' is equicontinuous on E, and therefore forms a precompact subset of E'_p. It follows from the above result that E'_p is trans-separable, and so the precompact subsets of E are metrisable. Note in particular that every DF space satisfies these two conditions, so we are able to deduce (as Pfister did) that the precompact subsets of a DF space are metrisable.

The condition that E be quasi-ℓ^∞-barrelled cannot be weakened to E being quasi-c_0-barrelled. To see this, let F be a non-separable reflexive Banach space, and suppose that E is the same space equipped with the topology of uniform convergence on the $\beta(F', F)$-precompact subsets of F'. Because E is a gDF space [5, 12.5.6], it is quasi-c_0-barrelled. The unit ball K of F is precompact in E, but it cannot be separable in E because it is not separable in F (separability is a duality invariant). This means that K is a precompact set in E that fails to be metrisable.

Note that a uniform space is precompact if and only if all its countable subsets are precompact. Combining the lemma and the theorem, we arrive at the following version of Cascales and Orihuela's result.

Corollary 2. *Let E be a locally convex space. If there exists a family $\{P(\alpha) : \alpha \in \mathcal{N}\}$ of precompact subsets of E'_p such that*

(1) $\bigcup_{\alpha \in \mathcal{N}} P(\alpha) = E'$, *and*

(2) $(\forall \alpha, \beta \in \mathcal{N})\ \alpha \leqslant \beta \Rightarrow P(\alpha) \subseteq P(\beta)$,

then the precompact subsets of E are metrisable. ∎

Let $\mathcal{C}(X)$ denote the space of all continuous scalar-valued functions on a completely regular Hausdorff space X. The Lemma above can be modified to show that if $\mathcal{C}(X)$ is trans-separable when equipped with its compact-open topology, then the compact subsets of X are metrisable (see [7, IV.1.2]). It is thus possible to formulate a result

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analogous to Corollary 2 concerning the metrisability of the precompact subsets of a general uniform space X.

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